

**CODIMENSION 1 AND 2 BIFURCATIONS
OF FIXED POINTS OF DIFFEOMORPHISMS
AND PERIODIC SOLUTIONS OF VECTOR FIELDS**

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RÉSUMÉ. Dans cet article on passe en revue les bifurcations de codimension 1 et 2 des points fixes des difféomorphismes. On met en évidence la relation entre les bifurcations des points fixes et les bifurcations des solutions périodiques des champs de vecteurs. L'étude des bifurcations des solutions périodiques se ramène à celle des bifurcations de points fixes de l'application de premier retour de Poincaré. D'autre part, la bifurcation de codimension 2 d'un point fixe avec deux valeurs propres $e^{\pm i\alpha}$ (resp. ± 1) s'étudie à l'aide des champs de vecteurs symétriques sous un groupe de rotation fini ou infini (resp. sous une symétrie axiale). On met l'emphasis sur la dérivation géométrique des résultats, ainsi que sur leur interprétation.

ABSTRACT. In this paper we review codimension 1 and 2 bifurcations of fixed points of diffeomorphisms. We show the relationship between bifurcations of fixed points and bifurcations of periodic solutions. First, the bifurcations of periodic solutions are studied by means of the fixed points of the Poincaré return map. Then the codimension 2 bifurcation of a fixed point with a pair of eigenvalues $e^{\pm i\alpha}$, (resp. ± 1) is studied by means of vector fields that are symmetric under finite or infinite rotation groups (under an axial symmetry). Emphasis is laid on geometric derivation and interpretation of results.

1. Introduction. In this paper we describe how a fixed point of a local diffeomorphism of $\mathbb{R}^n$ loses its stability. The simplest types of destabilization are via the appearance of other fixed or periodic points, or the emergence of an invariant attractive closed curve. We restrict ourselves here to loss of stability through codimension 1 and 2 bifurcations. Some of these results are well known to both pure and applied mathematicians, especially, for example, exchange of stability of two fixed points when they pass through each other and doubling period bifurcation. These appear for diffeomorphisms of $\mathbb{R}$, and, correspond to an eigenvalue crossing the unit circle at values 1 and -1 , respectively.

If we restrict ourselves to the case $n \geq 2$, we encounter Hopf bifurcation, which corresponds to loss of stability of a fixed point when two complex conjugate eigenvalues cross the unit circle at points $e^{\pm 2\pi i\alpha}$. We observe the emergence of an invariant closed curve on which the diffeomorphism is orientation-preserving. (In fact, it is close to a rotation through an angle $2\pi\alpha$). But orientation-preserving diffeomorphisms on the circle are in general not topologically equivalent. To study topological equivalence, we introduce an invariant: rotation number. Two topologically equivalent orientation-preserving diffeomorphisms must have the same rotation number. This is a sufficient condition in the case of an irrational rotation number. In the case of a rational rotation number p/q , the diffeomorphisms have periodic points. Generically they have a finite number of hyperbolic periodic points, in which case the dynamics is completely determined by the number of periodic points. Since rational and irrational numbers are both dense in $\mathbb{R}$, the two cases are inseparable. When eigenvalues cross the unit circle at points with an argument commensurable with 2π we will have open regions with rational rotation numbers. We call this phenomenon a resonance effect. As a result, the bifurcation corresponding to two eigenvalues crossing the unit circle is best studied as a 2-parameter bifurcation, one parameter

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for transversal crossing on the unit circle, and another for perturbation of the eigenvalues arguments. This approach is taken by several workers ([2], [13], [15], [18]).

In this paper we give an elementary, though fairly complete treatment of this phenomenon. Our main inspiration is the work of Arnold [2], who provides a beautiful geometrical insight. He studies bifurcations near resonances $e^{\pm 2\pi ip/q}$ by analyzing bifurcations of vector fields symmetric under rotation of angle $2\pi/q$. Here we derive this transformation in detail, since it is rather obscure in the literature. We go on to study bifurcations for different values of q , and reinterpret them in terms of bifurcations of the fixed points of the map. Among the resonance effects, we distinguish between weak ($q > 5$) and strong resonances ($q = 1, 2, 3, 4$). In weak resonances, Hopf bifurcation is the principal phenomenon, giving rise to an invariant attractive closed curve. The additional bifurcations are just modifications of the dynamics on the invariant attractive closed curve. In strong resonances, however, we find several fixed or periodic points emerging before Hopf bifurcation, and some of them are saddle points. This is followed by an interaction between the invariant closed curve that emerged in Hopf bifurcation and the stable and unstable manifolds of the other fixed or periodic points. This yields transversal crossings of the stable and unstable manifolds, implying chaotic behaviour. Work on the strong resonance of order 4 is not complete, and we mention the unresolved problems relating to this resonance.

In constructing bifurcation diagrams for vector fields symmetric under rotations, we give geometric evidence for the existence of bifurcation curves. We avoid long technical proofs, which would ensure us that the bifurcation diagram is complete. However, we discuss the interpretation in terms of bifurcations of fixed points in detail, since the treatment in the literature is incomplete.

Similarly, we analyze bifurcation of a fixed point with two eigenvalues ± 1 by studying a vector field with one axial symmetry, as first proposed by Zholondek [19]. In the latter case, as in the case of Hopf bifurcation, the description of the bifurcation is incomplete when the vector field has saddle connections, since this is normally interpreted by means of transversal homoclinic orbits accompanied by chaotic behaviour.

It has been known since Poincaré that the stability or bifurcations of periodic solutions of vector fields can be studied by analyzing the fixed points of the Poincaré return map. We therefore interpret all our results in both contexts. The Hopf bifurcation theorem for a fixed point of the Poincaré map when the eigenvalues take values $e^{\pm 2\pi i\alpha}$ corresponds to a bifurcation to a 2-torus when the limit cycle loses its stability. When the system is linear at the bifurcation point, the trajectories are quasi-periodic if α is irrational (periodic if α is rational). The non-linear case is measuring how qualitatively different the system is from the underlying linear system and determining which quasi-periodic (periodic) motions of the linear case remain in the non-linear case.

In general, we study the stability of a fixed point of an application by means of the eigenvalues of its linear parts:

- If all eigenvalues λ satisfy $|\lambda| < 1$ (resp. $|\lambda| > 1$), then the fixed point is asymptotically stable (resp. unstable).
- If all eigenvalues λ satisfy $|\lambda| \neq 1$, then the fixed point is hyperbolic, i.e. structurally stable. It can be asymptotically stable, unstable, or a saddle point.
- If some of the eigenvalues satisfy $|\lambda| = 1$, the stability of the fixed point has to be checked using higher order terms. Moreover, the fixed point is structurally unstable, and small perturbations may generate bifurcations.

Here we are interested in studying the simplest bifurcations of fixed points of local diffeomorphisms, which requires that we construct universal unfolding of singularities. A

universal unfolding of a local diffeomorphism F "contains" all behaviour appearing in any perturbation of the local diffeomorphism. Given a local diffeomorphism F with a non-hyperbolic fixed point at the origin, we embed F in a family F_ε depending of k parameters $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ such that for any perturbation G_μ of F depending on m parameters $(\mu_1, \dots, \mu_m)$, any member of G_μ is topologically equivalent to an F_ε . The precise definition will be given in Section 2. The minimum number of parameters needed for a universal unfolding is called the *codimension* of the singularity.

The intuitive notion of codimension is as follows. If a singularity of codimension greater than k appears in a k -parameter family of diffeomorphisms, then it is possible to remove the singularity by a small perturbation of the family. Singularities of codimension $\leq k$ are the only unavoidable singularities in a k -parameter family [2].

According to the centre manifold theorem, we can search for universal unfoldings in a space of a dimension equal to the number of unit-modulus eigenvalues. More precisely, if the local diffeomorphism F on $\mathbb{R}^n$ has n eigenvalues on the unit circle, then its dynamics is a direct product of the dynamics of a map f on $\mathbb{R}^{n_0}$, with a map $(x_+, x_-) \mapsto (2x_+, \frac{1}{2}x_-)$ (in the case of an orientation preserving diffeomorphism). The precise form of the centre manifold theorem will be given in Section 2.

The simplest way for a fixed point to lose its stability is through a real eigenvalue 1 or -1 , which is the case inside a centre manifold of dimension 1. Under a non-degeneracy condition we get bifurcation of codimension 1. Under additional degeneracy we have codimension 2 singularities. Due to resonance effects, the least degenerate singularity with two eigenvalues on the unit circle is of codimension 2. The last codimension 2 singularity is the case of two eigenvalues 1 and -1 .

This paper is organized in the following way. In the second section we briefly recall the definitions of universal unfolding and normal forms in the neighborhood of a fixed point of a diffeomorphism, and state the centre manifold theorem. In the third section we discuss bifurcations with one eigenvalue equal to 1; in the fourth section, bifurcations with one eigenvalue -1 ; and in the fifth section, bifurcations with a pair of eigenvalues on the unit circle. In the last section we analyze the bifurcation for a pair of eigenvalues 1 and -1 .

REMARK. For simplicity we will always consider C^∞ diffeomorphisms or vector fields. This allows us to use the very powerful tool of the Malgrange-Weierstrass preparation theorem. Of course the following treatment is also valid under weaker hypotheses. In general, if our treatment involves terms of the normal form up to order n , it is valid for diffeomorphisms or vector fields of class C^{n+1} .

2. Universal unfolding, normal forms, and the centre manifold theorem. We first give the precise definition of universal unfolding of a fixed point of diffeomorphism.

DEFINITION [2]. Let $F: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a local diffeomorphism of class C^r . A deformation F_ε of class C^r , depending on $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ such $F_\varepsilon|_{\varepsilon=0} = F$, is a *universal unfolding* (or *versal deformation*) of F if and only if, given any perturbation G_μ of class C^r depending on $\mu = (\mu_1, \dots, \mu_m)$ such that $G_\mu|_{\mu=0} = F$,

- i) there exists neighborhoods U of 0 in ε -space, V of 0 in μ -space, W of 0 in $\mathbb{R}^n$,
- ii) there exists a function $\phi: V \rightarrow U$,
- iii) there exists a map $H: W \times V \rightarrow W$ such that, for fixed μ , $H_\mu = H(-, \mu)$ is a homeomorphism, and for each μ , G_μ is topologically equivalent to $F_{\phi(\mu)}$ through H_μ , i.e. $H_\mu^{-1}G_\mu H_\mu = F_{\phi(\mu)}$.

The normal forms serve as a tool to construct universal unfoldings. The diffeomorphism

is first brought to normal form, and the missing terms in the normal form usually suggest a universal unfolding. We now proceed to discuss the normal form (following [2]).

Consider a local diffeomorphism

$$F: (\mathbb{R}^n, O) \rightarrow (\mathbb{R}^n, 0), \quad (2.1)$$

whose linear part has eigenvalues on the unit circle. Suppose that

$$F(x) = Ax + F_r(x) + F_{r+1}(x) + \dots, \quad (2.2)$$

where $F_r(x)$ is homogeneous of degree r . We introduce a change of coordinates

$$x = G(y) = y + G_r(y), \quad (2.3)$$

in order to eliminate terms of degree r . This gives the equation

$$\begin{aligned} G^{-1}FG(y) &= G^{-1}(Ay + AG_r(y) + F_r(y + G_r(y)) + \dots) \\ &= Ay + AG_r(y) + F_r(y + G_r(y)) - G_r(Ay + AG_r(y) + F_r(y + G_r(y))) + \dots \end{aligned} \quad (2.4)$$

Comparing this with

$$G^{-1}FG(y) = Ay + O(|y|^{r+1}) \quad (2.5)$$

we get the equation for the terms of degree r :

$$F_r(y) = G_r(Ay) - AG_r(y) = M_A(G_r(y)), \quad (2.6)$$

which we call the *homological equation*. At each step (i.e. for each r) we can eliminate the image of M_A . M_A is a linear operator on the set of homogeneous polynomial vectors of degree r . The case where A is a diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ can be described simply. If $G_r(y) = y^m e_s = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n} e_s$, where e_s is the s^{th} vector of the standard basis of $\mathbb{R}^n$, then

$$M_A(G_r(y)) = (\lambda_1^{m_1} \lambda_2^{m_2} \dots \lambda_n^{m_n} - \lambda_s) G_r(y). \quad (2.7)$$

Therefore the image of M_A contains all *non-resonant* terms, i.e. all terms for which $\lambda_1^{m_1} \lambda_2^{m_2} \dots \lambda_n^{m_n} - \lambda_s \neq 0$. The normal form is now an expression with resonant terms only.

If we consider the following particular cases, we can arrive at the normal forms:

- One eigenvalue $\lambda = 1$. All terms are resonant. The normal form is:

$$F(x) = x + a_2 x^2 + a_3 x^3 + \dots \quad (2.8)$$

- One eigenvalue $\lambda = -1$. We have the resonances $\lambda = \lambda^3 = \lambda^5 = \dots$, and the normal form:

$$F(x) = -x + a_3 x^3 + a_5 x^5 + \dots \quad (2.9)$$

- One pair of eigenvalues $\lambda_{1,2} = e^{\pm 2\pi i \alpha}$. In this case we obtain the resonances $\lambda_1 = \lambda_1^2 \lambda_2 = \lambda_1^3 \lambda_2^2 = \dots$, $\lambda_2 = \lambda_2^2 \lambda_1 = \lambda_2^3 \lambda_1^2 = \dots$. These are the only resonances if α is irrational. The case $\alpha = p/q$ yields the extra resonances $\lambda_1 = \lambda_2^{q-1} = \dots$. The general resonances thus are $\lambda_1 = \lambda_1^{m_1} \lambda_2^{m_2}$ where $m_1 - m_2 = 1 \pmod{q}$, and similarly

for λ_2 . Here the normal form is most simply expressed in complex coordinates. For irrational α it is given by:

$$F(z) = e^{2\pi i \alpha} z + c_1 z^2 \bar{z} + c_2 z^3 \bar{z}^2 + \dots \quad (2.10)$$

In the case of rational $\alpha = p/q$, we get

$$F(z) = e^{2\pi i p/q} z + c_1 z^2 \bar{z} + c_2 z^3 \bar{z}^2 + \dots + A \bar{z}^{q-1} + O(|z|^q). \quad (2.11)$$

— One pair of eigenvalues ± 1 . We have the resonances

$$\begin{aligned} \lambda_1 = \lambda_1^2 = \lambda_1^3 = \dots = \lambda_1 \lambda_2^2 = \lambda_1 \lambda_2^4 = \dots \lambda_2^2 = \lambda_2^4 = \dots, \\ \lambda_2 = \lambda_2^3 = \lambda_2^5 = \dots = \lambda_1 \lambda_2 = \lambda_1^2 \lambda_2 = \dots \end{aligned} \quad (2.12)$$

The normal form is then:

$$\begin{aligned} F(x_1, x_2) = (x_1 + a_2 x_1^2 + a_3 x_1^3 + b_1 x_1 x_2^2 + b_2 x_1 x_2^4 + e_1 x_2^2 + e_2 x_2^4 + \dots, \\ -x_2 + c_1 x_2^3 + c_2 x_2^5 + d_1 x_1 x_2 + d_2 x_1^2 x_2 + \dots). \end{aligned} \quad (2.13)$$

REMARK. (2.10) and (2.11) are the forms (up to higher order terms) of an equivariant map under all rotations for α irrational (under rotation of angle $2\pi/q$ for $\alpha = p/q$). (2.13) is the form of a map symmetric about the x_1 -axis.

Since we are looking for equivalent classes of diffeomorphisms, it is natural to start with a diffeomorphism already in normal form, up to a certain degree.

We now give the most general form of the centre manifold theorem.

THEOREM [6]. Let $F: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a diffeomorphism of class C^r with a fixed point at the origin, and suppose that the linear part has

- i) n_+ eigenvalues greater than 1 in modulus, with eigenspace E^{n_+} ,
- ii) n_- eigenvalues less than 1 in modulus, with eigenspace E^{n_-} ,
- iii) n_0 eigenvalues with unit modulus, and eigenspace E^{n_0} .

Then, in a neighborhood of the origin, there exist invariant C^r -manifolds W^u , W^s and W^0 tangent to E^{n_+} , E^{n_-} and E^{n_0} respectively. W^u and W^s are unique, while W^0 is not necessarily unique. Moreover, there exists a change of coordinates in which the map becomes

$$F(x_+, x_-, x_0) = (2x_+, \frac{1}{2}x_-, f(x_0)), \quad (2.14)$$

for the case when $F|_{E^{n_+}}$ and $F|_{E^{n_-}}$ are orientation preserving. We change the sign of one coordinate in the natural way for the other cases.

Bifurcation with the eigenvalue $\lambda = 1$. The first codimension 1 case is called *saddle-node bifurcation*, and corresponds to a map $F: \mathbb{R} \rightarrow \mathbb{R}$ with normal form:

$$F(x) = x + ax^2 + O(x^3), \quad a \neq 0 \quad (3.1)$$

In this case a universal unfolding is given by:

$$F_\epsilon(x) = F(x) + \epsilon = x + \epsilon + ax^2 + O(x^3). \quad (3.2)$$

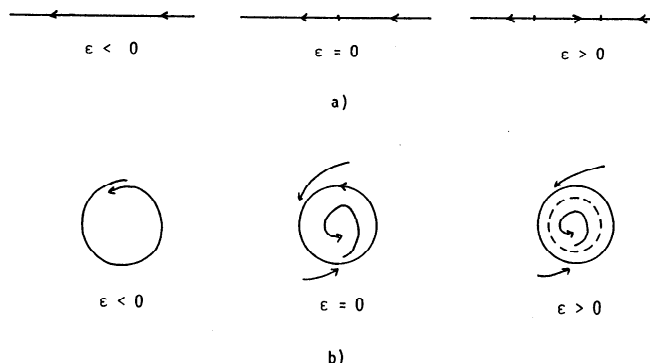


Figure 1. Saddle-node bifurcation in the case $a < 0$: a) fixed point, b) periodic orbit.

By the Malgrange-Weierstrass preparation theorem (see Appendix), the equation for fixed points $F_\epsilon(x) = x$ gives a maximum of two fixed points: $F_\epsilon(x) - x = (x^2 + b(\epsilon)x + c(\epsilon))H(x, \epsilon)$ in a neighborhood of the origin, where $b(0) = c(0) = 0$ and $H(0, 0) \neq 0$. The equation for fixed points is $x^2 + b(\epsilon)x + c(\epsilon) = 0$ which can be approximated by $ax^2 + \epsilon = 0$. This means we have two fixed points for $a\epsilon < 0$, no fixed point for $a\epsilon > 0$, and a double fixed point for $\epsilon = 0$ (Figure 1a and 2a).

In the case of bifurcations of limit cycles this corresponds to the coalescence of two limit cycles before they disappear (Figure 1b).

Depending on the constraints of the system, some significant variations may arise:

- If the origin always remains a fixed point we have a *transcritical bifurcation*. Here, we consider the following universal unfolding of (3.1):

$$F_\epsilon(x) = x(1 + \epsilon) + ax^2 + O(x^3), \quad (3.3)$$

which always has two fixed points that pass through each other and exchange stability (Figure 2b).

- If the function is odd, due to a symmetry of the system, the *pitchfork bifurcation* occurs. This is characterized by the following universal unfolding (see Figure 2c):

$$F_\epsilon(x) = x(1 + \epsilon) + ax^3 + O(x^5), \quad a \neq 0. \quad (3.4)$$

We now consider the similar case of codimension 2, corresponding to a map with normal form:

$$F(x) = x + ax^3 + O(x^4). \quad (3.5)$$

A universal unfolding is given by:

$$F_\epsilon(x) = \epsilon_1 + x + \epsilon_2 x + ax^3 + O(x^4). \quad (3.6)$$

As before, the equation for fixed points can be approximated by $ax^3 + \epsilon_2 x + \epsilon_1 = 0$, which has three zeros for $4a\epsilon_2^3 + 27a^2\epsilon_1^2 > 0$, one for $4a\epsilon_2^3 + 27a^2\epsilon_1^2 < 0$ (and two zeros — a simple and a double — on the curve $4a\epsilon_2^3 + 27a^2\epsilon_1^2 = 0$) See Figure 3a for the fixed points and Figure 3b for the case of three limit cycles.

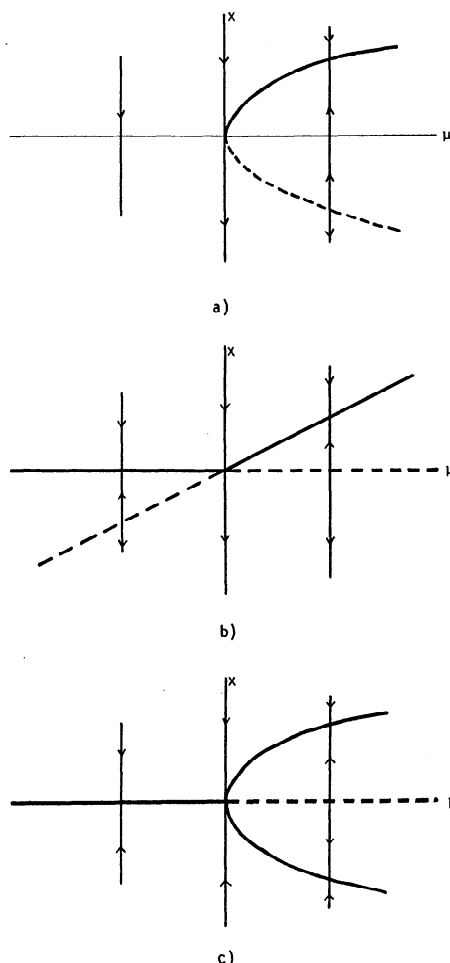


Figure 2. Codimension 1 bifurcation in the case of the eigenvalue 1: a) saddle-node bifurcation, b) transcritical bifurcation, c) pitchfork bifurcation.

4. Bifurcation with eigenvalue -1 . The normal form of the system here is

$$F(x) = -x + a_3x^3 + a_5x^5 + O(x^7). \quad (4.1)$$

Bifurcation of codimension 1, called *period doubling bifurcation* (or *flip bifurcation*), corresponds to $a_3 \neq 0$. The origin is stable if and only if $a_3 > 0$. To find the unfolding, we note that (by the implicit function theorem) any perturbation of (4.1) will still have a unique fixed point, but allows the eigenvalue to vary from -1

$$F_\epsilon(x) = -(1 + \epsilon)x + a_3x^3 + O(x^5). \quad (4.2)$$

The second iterate of the map

$$F_\epsilon^2(x) = (1 + 2\epsilon + \epsilon^2)x + (-2 - 4\epsilon - 3\epsilon^2 - \epsilon^3)a_3x^3 + O(x^5), \quad (4.3)$$

always has a fixed point at 0 (which is already the fixed point of F). As in Section 3, we get two more fixed points when $\epsilon a_3 > 0$. These fixed points are stable (unstable) when $a_3 > 0$ ($a_3 < 0$). The fixed points of F_ϵ^2 correspond to period 2 points of F_ϵ (see Figure 4a).

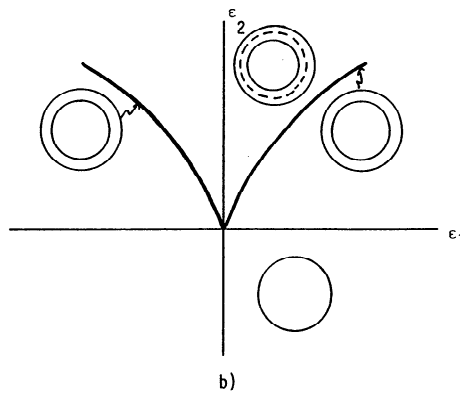
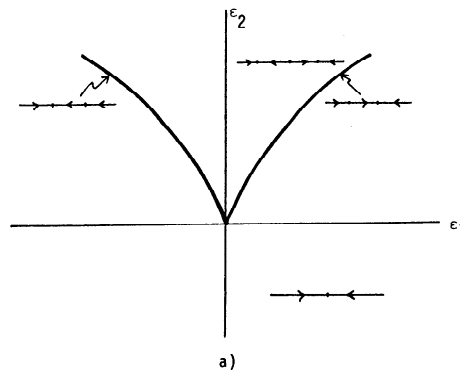


Figure 3. Codimension 2 bifurcation for the eigenvalue 1: a) fixed point, b) periodic orbit.

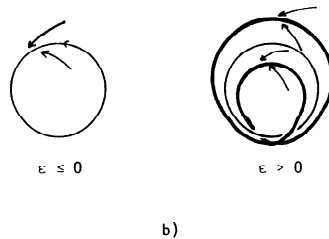
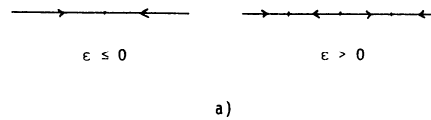


Figure 4. Codimension 1 bifurcation with eigenvalue -1 : a) fixed point, b) periodic orbit.

For bifurcation of limit cycles, this corresponds to a loss (or gain) of stability of the limit cycle and the appearance of a stable (unstable) limit cycle of twice the period (see Figure 4b). Note that although the centre manifold of the map is of dimension one, this

phenomenon cannot happen in the plane and occurs only in $\mathbb{R}^n$ for $n \geq 3$, since the centre manifold for the limit cycle is locally a Möbius strip (trajectories cannot intersect in the case of a vector field).

The codimension 2 case corresponds to $a_3 = 0$ and $a_5 \neq 0$, with unfolding

$$F_\varepsilon(x) = -(1 + \varepsilon_1)x + \varepsilon_2 x^3 + a_5 x^5 + O(x^7). \quad (4.4)$$

The equation for fixed points of F_ε^2 is now given by

$$(1 + \varepsilon_1)^2 x - \varepsilon_2 [1 + \varepsilon_1 + (1 + \varepsilon_1)^3] x^3 - (1 + \varepsilon_1) [a_5 (1 + (1 + \varepsilon_1)^4 - 3\varepsilon_2^2 (1 + \varepsilon_1)^2) x^5 + O(x^7)] = x. \quad (4.5)$$

and the regions in parameter-space with hyperbolic fixed points can be approximated by

i)

$$\varepsilon_2^2 + 4\varepsilon_1 a_5 > 0, \quad \varepsilon_1 a_5 < 0, \quad \varepsilon_2 a_5 < 0, \quad (4.6)$$

for four nonzero fixed points.

ii)

$$\varepsilon_2^2 + 4\varepsilon_1 a_5 > 0, \quad \varepsilon_1 a_5 > 0, \quad (4.7)$$

for two nonzero fixed points.

iii)

$$\varepsilon_2^2 + 4\varepsilon_1 a_5 > 0, \quad \varepsilon_1 a_5 < 0, \quad \varepsilon_2 a_5 > 0, \quad (4.8)$$

or

$$\varepsilon_2^2 + 4\varepsilon_1 a_5 < 0, \quad (4.9)$$

when there are no nonzero fixed points.

The line $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$, corresponds to the period doubling bifurcation, while the curve approximated by $\varepsilon_2^2 + 4\varepsilon_1 a_5 = 0$, $\varepsilon_1 a_5 < 0$, $\varepsilon_2 a_5 < 0$, corresponds to the coalescence of two pairs of period 2 points of F_ε (cf. Figure 5a).

In the case of vector fields we have the same bifurcation diagram. The case with no (resp. two, four) nonzero periodic points corresponds to zero (resp. one, two) period 2 limit cycles located in the centre manifold. When there are two limit cycles, one is attractive and the other is repulsive (cf. Figure 5b).

REMARK. The bifurcation diagram in this case is reminiscent of the bifurcation diagram for Hopf bifurcation of order 2.

We have computed (using Macsyma) the coefficients a_3 and a_5 of the normal form of a map

$$F(x) = -x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \dots \quad (4.10)$$

These are given by

$$a_3 = b_3 + b_2^2, \quad (4.11)$$

$$a_5 = b_5 + 3b_2 b_4 + \frac{13}{4} b_2^2 b_3 + \frac{5}{4} b_2^4. \quad (4.12)$$

5. Bifurcation with a pair of eigenvalues $e^{\pm 2\pi i \alpha}$. Using the implicit function theorem, we note that any perturbation of the map has an unique fixed point (as the case of eigenvalue -1). In the case of an irrational α or a rational $\alpha = p/q$, with q large enough ($q \geq 5$), it seems natural to take

$$F_\varepsilon(z) = (1 + \varepsilon) e^{2\pi i \alpha} z + c_1 z^2 \bar{z} + O(|z|^4) \quad (5.1)$$

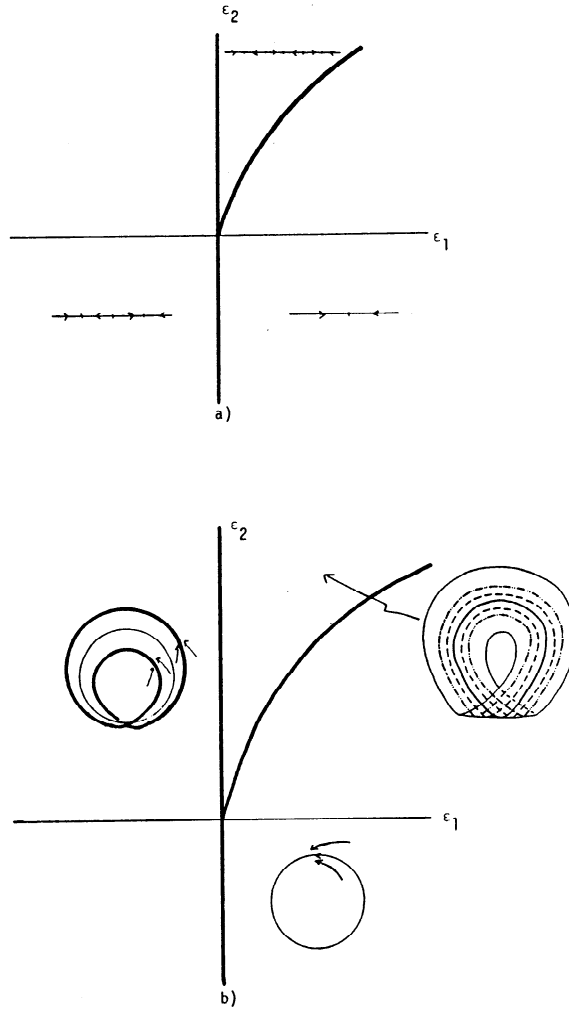


Figure 5. Codimension 2 bifurcation with eigenvalue -1 : a) fixed point, b) periodic orbit.

as a universal unfolding of (2.10) or (2.11), under the hypothesis $\operatorname{Re}(\bar{c}_1 e^{2\pi i \alpha}) \neq 0$, the meaning of which we will discuss later. We then apply the Hopf theorem for diffeomorphisms [13], and obtain an attractive or repulsive invariant closed curve.

The underlying idea of the Hopf theorem for diffeomorphisms is this: in polar coordinates, $z = r e^{i\theta}$, $F_\epsilon(z) = R e^{i\Phi}$, (5.1) gives

$$\begin{aligned} R &= r [(1 + \epsilon) + r^2 \operatorname{Re}(\bar{c}_1 e^{2\pi i \alpha}) + O(r^3)] \\ \Phi &= 2\pi\alpha + \theta + O(r). \end{aligned} \quad (5.2)$$

For small r , Φ is given approximately by $2\pi\alpha + \theta$. Suppose first that $O(r^3)$ is independent of θ in the expression of R . An invariant closed curve is thus a circle satisfying the equation $R = r$. Solving $R^2 = r^2$ by the implicit function theorem, we find a unique solution $r^2 = r^2(\epsilon)$, provided $\operatorname{Re}(\bar{c}_1 e^{2\pi i \alpha}) \neq 0$. This gives an invariant closed curve for $\epsilon \operatorname{Re}(\bar{c}_1 e^{2\pi i \alpha}) < 0$. The closed curve is attractive (repulsive) if $\operatorname{Re}(\bar{c}_1 e^{2\pi i \alpha}) < 0$ (> 0). The result is the same when $O(r^3)$ is dependent of θ , except that the invariant closed curve

is no longer a circle (the proof is far more complex). We state the theorem in a simple form. The more general form can be found in [13], together with a complete proof.

THEOREM. Assume the function F in (5.1) is of class C^k , for $k \geq 6$, and $\operatorname{Re}(\overline{c_1}e^{2\pi i\alpha}) < 0$ (resp. > 0). Then there exists an invariant attractive (resp. repulsive) closed curve $r = r(\varepsilon, \theta)$, bifurcating from 0, for small ε satisfying $\varepsilon \operatorname{Re}(\overline{c_1}e^{2\pi i\alpha}) < 0$.

REMARKS.

- (1) The condition $\operatorname{Re}(\overline{c_1}e^{2\pi i\alpha}) \neq 0$ has a geometric interpretation. It is equivalent to a nonzero scalar product $\langle c_1, e^{2\pi i\alpha} \rangle$, where the complex numbers c_1 and $e^{2\pi i\alpha}$ are considered as vectors of $\mathbb{C} = \mathbb{R}^2$. When $\langle c_1, e^{2\pi i\alpha} \rangle = 0$, the cubic term $c_1 z^2 \bar{z}$ has no radial contribution, and the stability is determined by higher-order terms.
- (2) The inverse function theorem applied at the origin ensures that for sufficiently small ε , the map restricted to the invariant closed curve is a diffeomorphism.

In the Hopf bifurcation theorem for vector fields, an unfolding similar to (5.1) is universal, since vector fields having a unique attractive (repulsive) limit cycle with same orientation are topologically equivalent. But this does not hold for diffeomorphisms, since orientation-preserving diffeomorphisms of the circle (e.g., the diffeomorphism restricted to the invariant closed curve) are in general not topologically equivalent. Diffeomorphisms with periodic points cannot be equivalent to diffeomorphisms with none. There is an invariant for topologically equivalent diffeomorphisms of the circle: the rotation number, and it can also serve as a tool for detecting periodic points.

DEFINITION. Let $f: S^1 \rightarrow S^1$ be a diffeomorphism. Then f can be represented by a map $\phi: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\phi(x+1) = \phi(x) + 1$. The rotation number is defined as

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{\phi^n(x)}{n}. \quad (5.3)$$

PROPERTIES OF ROTATION NUMBER.

- i) $\rho(f)$ is independent of the point x used in the definition.
- ii) $\rho(f)$ is rational and equal to p/q if and only if f has a periodic point of period q .
- iii) If $\rho(f) = \alpha$ is irrational, then f is topologically equivalent to a rotation of angle $2\pi\alpha$. (This is Denjoy's theorem [2].)

From the previous remarks it follows that (5.1) is an unfolding, but not a universal unfolding. In a universal unfolding of (2.10), we would expect to find diffeomorphisms with irrational rotation number, and in the universal unfolding of (2.11), diffeomorphisms with periodic points of period q should appear. Since both the rational and the irrational numbers are dense in $\mathbb{R}$, it is natural for a universal unfolding to depend on two parameters

$$F_\varepsilon(z) = (1 + \varepsilon_1 + i\varepsilon_2)e^{2\pi i\alpha}z + c_1 z^2 \bar{z} + O(|z|^4), \quad (5.4)$$

at least when $\alpha \neq p/q$ for $q \leq 4$.

We could study (5.4) in more detail. However, we will adopt another approach. We start by studying the bifurcation of a periodic orbit for which the Poincaré map has eigenvalues $e^{\pm 2\pi i\alpha}$. We show how this can be analyzed by means of a planar vector field symmetric under rotations of angle $2\pi\alpha$. We then show that (5.4) itself can be further studied by analyzing a two-parameter deformation of a symmetric vector field.

We begin with a vector field having a periodic solution corresponding to a fixed point of the Poincaré map on a cross-section, and suppose that the linearized Poincaré map has eigenvalues $e^{\pm 2\pi i\alpha}$: i.e. we consider a vector field in $\mathbb{R}^3$. We transform the vector field in the following steps:

- i) We localize the vector field at the periodic solution, which yields a system with periodic coefficients. We then apply Floquet theory to the linear part.
- ii) We bring the system to Poincaré-Birkhoff-Dulac normal form, in "toroidal" coordinates $(t, z, \bar{z})$ in the neighborhood of the periodic solution.
- iii) We change the coordinates $(z, \bar{z})$ to coordinates $(\zeta, \bar{\zeta})$ spiralling along t , so that the system becomes independent of t . The system in $(\zeta, \bar{\zeta})$ is invariant under rotations of angle $2\pi\alpha$ to arbitrary large order.
- i) When we localize the vector field at the periodic solution, we obtain an equation with 2π -periodic coefficients in t .

$$\dot{z} = A(t) + f(z, \bar{z}, t) \quad f(z, \bar{z}, t) = O(|z|^2). \quad (5.5)$$

The monodromy operator (a diagonal map with eigenvalues $e^{\pm 2\pi i\alpha}$) now coincides with the Poincaré map. Using Floquet theory, (by a linear 2π -periodic change of coordinates) the linear part of (5.5) can be brought to

$$\dot{z} = i\alpha z. \quad (5.6)$$

This change of coordinates brings the whole equation to

$$\dot{z} = i\alpha z + g(z, \bar{z}, t) \quad g(z, \bar{z}, t) = O(|z|^2) \quad g(z, \bar{z}, t + 2\pi) = g(z, \bar{z}, t), \quad (5.7)$$

where $g = \sum a_{m,k} z^{m_1} \bar{z}^{m_2} e^{ikt} = g_r(z, \bar{z}, t) + O(|z|^{r+1})$.

- ii) Following the example of the normal form for vector fields we perform a change of variables

$$z = w + \sum b_{m,k} w^{m_1} \bar{w}^{m_2} e^{ikt} = w + h_r(w, \bar{w}, t) + O(|w|^{r+1}), \quad (5.8)$$

in order to eliminate as many terms in the second member of (5.7) as possible. This procedure is done degree by degree [2]. The normal form obtained is called the *Poincaré-Birkhoff-Dulac normal form*. To eliminate homogeneous terms of degree r in (5.7), we take (5.8) with $|m| = m_1 + m_2 = r$. In order to transform the equation to the form

$$\dot{w} = i\alpha w + O(|w|^{r+1}), \quad (5.9)$$

we must solve the homological equation

$$g_r = i\alpha w \frac{\partial h_r}{\partial w} - i\alpha \bar{w} \frac{\partial h_r}{\partial \bar{w}} + \frac{\partial h_r}{\partial t} - i\alpha h_r = L_\alpha(h_r). \quad (5.10)$$

An h_r of the form $h_r = w^{m_1} \bar{w}^{m_2} e^{ikt}$ is an eigenvector with eigenvalue $\lambda = ik + (m_1 + m_2 - 1)\alpha$. Then h_r is in the image of L_α if and only if $\lambda \neq 0$. The resonant terms which appear in the normal form are those for which $\lambda = 0$. In the case of irrational α this gives

$$m_1 = m_2 + 1, \quad k = 0. \quad (5.11)$$

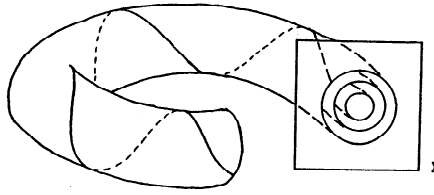


Figure 6. Trajectories of quasi-periodic or periodic flow in the neighborhood of a periodic solution.

In case of rational $\alpha = p/q$ this gives $kq + (m_1 + m_2 - 1)p = 0$ which gives

$$k = ps, \quad m_1 + m_2 - 1 = -qs \quad \text{for } s \in \mathbb{Z}. \quad (5.12)$$

For terms in $w, \bar{w}$, this gives

$$m_1 - m_2 - 1 = 0 \pmod{q}. \quad (5.13)$$

The resonant term of lowest degree satisfying $m_1 - m_2 - 1 = -qs \neq 0$ is the term $\bar{w}^{q-1}$. Replacing w by z , our normal form is then, to arbitrarily large order r

$$\dot{z} = i\alpha z + \sum_{m_1 - m_2 - 1 = 0(q)} a_{m,k} z^{m_1} \bar{z}^{m_2} e^{ikt} + O(|z|^{r+1}), \quad (5.14)$$

where it is understood that the sum is over the set $m_1 - m_2 - 1 = 0$ if α is irrational.

- iii) We change the coordinates $(t, z, \bar{z})$ to $(t, \zeta, \bar{\zeta})$ with $z = e^{i\alpha t} \zeta$. The system (5.14) becomes

$$\dot{\zeta} = \sum_{\substack{m_1 - m_2 - 1 = 0(q) \\ 2 \leq |m| \leq r}} a_{m,k} \zeta^{m_1} \bar{\zeta}^{m_2} + O(|\zeta|^{r+1}). \quad (5.15)$$

REMARKS.

- (1) If a vector field is symmetric under the rotation of angle $2\pi\alpha$, then it is symmetric under rotations of any multiple of $2\pi\alpha$. For α irrational (rational), the vector field is symmetric under a group of rotations with angles dense in the circle, and hence under any rotation (under the rotation of angle $2\pi/q$). Equation (5.15) is precisely the form of a system with vanishing linear part whose r -jet is symmetric under all rotations (rotations of angle $2\pi/q$) for α irrational (rational).
- (2) The same procedure could have been applied to a 2-parameter perturbation

$$\dot{z} = (\mu_1 + i\mu_2)z + i\alpha z + g(z, \bar{z}, t) \quad (5.16)$$

of (5.7), and would have given a 2-parameter perturbation of (5.15)

$$\dot{\zeta} = (\varepsilon_1 + i\varepsilon_2)\zeta + \sum_{\substack{m_1 - m_2 - 1 = 0(q) \\ 2 \leq |m| \leq r}} a_{m,k} \zeta^{m_1} \bar{\zeta}^{m_2} + O(|\zeta|^{r+1}). \quad (5.17)$$

- (3) We want to describe the geometry underlying the construction in i)-iii) in more detail, i.e. determine how different the flow of the vector field is from the quasi-periodic (or periodic if α is rational) flow with rotation number α (Figure 6). This flow is given by

$$\dot{z} = i\alpha z, \quad \dot{t} = 1, \quad (5.18)$$

and in ζ -coordinates by $\dot{\zeta} = 0$. We can imagine the ζ -coordinate as a complex coordinate in a section Σ at $t = 0$, transversal to the limit cycle. The change of coordinates from z -coordinates to ζ -coordinates can be viewed as the transportation of the vector field at the point (z, t) , back to the section Σ along the flow lines of the quasi-periodic flow. Here the vector field in ζ is independent of t because the system is in Poincaré-Birkhoff-Dulac normal form.

- (4) If we had applied the transformation $z = e^{i\alpha t}\zeta$, before bringing the system to normal form, the system in the ζ -coordinates would be non-autonomous

$$\dot{\zeta} = \sum a_{m,k} \zeta^{m_1} \bar{\zeta}^{m_2} e^{(\alpha(m_1-m_2-1)+k)it} + O(|\zeta|^{r+1}). \quad (5.19)$$

By taking the mean value for $t \in (-\infty, \infty)$ if α is irrational ($t \in (0, 2\pi q)$ for α rational), we get, as before, a system that is symmetric under all rotations (rotation of angle $2\pi/q$).

Consequently, the normal form eliminates contributions whose mean value is zero, and keeps all other contributions.

We now come to the problem of how (5.4) can be analyzed by means of a symmetric vector field. To do this, we show that there exists a vector field which is symmetric under all rotations for α irrational (under rotation of angle $2\pi/q$ for $\alpha = p/q$), such that the flow map at time 2π is equal to F_ε^q , to arbitrarily large order. We could have found a vector field with flow map equal to F_ε , to arbitrarily large order, but this is of no interest, since periodic points of the map will not be detected on a flow map. However, fixed points of the map F_ε^q will correspond to equilibrium points of the vector field (5.20) and periodic solutions of (4.5). We thus look for a vector field in $\mathbb{R}^2 \times S^1 \subset \mathbb{R}^3$:

$$\dot{z} = \beta z + \sum_{m_1-m_2-1=0(q)} a_m z^{m_1} \bar{z}^{m_2} + O(|z|^{r+1}) \quad \dot{t} = 1 \quad (5.20)$$

whose flow map at time 2π has the form

$$F_\varepsilon^q(z) = (1 + \varepsilon_1 + i\varepsilon_2)z + \sum_{m_1-m_2-1=0(q)} b_m z^{m_1} \bar{z}^{m_2} + O(|z|^{r+1}), \quad (5.21)$$

(with $q = 0$ when α is irrational). The vector field (5.20) has the periodic solution $z = 0$ in the "toroidal coordinates" $z, \bar{z}, t$. A determination of (5.20) is accomplished by identifying $\Phi(2\pi, z_0)$ and $F_\varepsilon^q(z_0)$ degree by degree, where $\Phi(t, z_0)$ is the solution of (5.20) with initial condition $\Phi(0, z_0) = z_0$. This solution has the same symmetry as the system and therefore the form

$$\Phi(t, z_0) = d_1(t)z_0 + \sum_{m_1-m_2-1=0(q)} d_m(t)z_0^{m_1} \bar{z}_0^{m_2} + O(|z_0|^{r+1}). \quad (5.22)$$

$d_1(t)$ and the d_m 's satisfy the differential equations

$$d_1'(t) = \beta d_1(t), \quad d_1(0) = 1, \quad (5.23)$$

$$d_3'(t) = \beta d_3(t) + a_{2,1}d_1(t)|d_1(t)|^2, \quad d_3(0) = 0, \quad \dots \quad (5.24)$$

These equations have a unique solution for the d_m 's as functions of β and the a_m 's. We next identify $\Phi(2\pi, z_0)$ with $F_\varepsilon^q(z_0)$. This gives a solution for β and the a_m 's as functions of $\varepsilon_1, \varepsilon_2$, and the b_m 's.

We have now reduced the study of bifurcations of fixed points, when the linear part has eigenvalues $e^{\pm 2\pi i \alpha}$, to the study of bifurcations of vector fields with zero linear part, which are symmetric under all rotations for α irrational (rotations of angle $2\pi/q$ if $\alpha = p/q$). We analyze bifurcations of the following vector field, which is symmetric under a rotation of angle $2\pi/q$

$$\dot{z} = (\varepsilon_1 + i\varepsilon_2) + c_1 z^2 \bar{z} + c_2 z^3 \bar{z}^2 + \dots + A \bar{z}^{q-1} + O(|z|^q), \quad (5.25)$$

with the non-degeneracy conditions $\text{Re}(c_1) \neq 0$, $A \neq 0$.

In this study we distinguish two cases: $q \geq 5$ and $q \leq 4$. The former is called *weak resonance*, and the latter, *strong resonance*. The distinction stems from the fact that, for $q \geq 5$, the term $A \bar{z}^{q-1}$ is small compared to $c_1 z^2 \bar{z}$. Then the dynamics is governed mainly by the term $c_1 z^2 \bar{z}$, which produces a limit cycle. The term $A \bar{z}^{q-1}$ only changes the dynamics on the limit cycle, with the appearance of $2q$ equilibrium points. However, in the case of strong resonances, the $A \bar{z}^{q-1}$ either becomes dominant over the term $c_1 z^2 \bar{z}$, in case $q \leq 3$, or is of equal importance in the case $q = 4$. This allows the emergence of additional equilibrium points before Hopf bifurcation, leading to an interaction between the limit cycle that emerged in Hopf bifurcation and the other singular points, usually in the form of homoclinic or heteroclinic connection.

THEOREM [2] [15]. *The bifurcation diagram of equation (5.25) in the case of weak resonances is given in Figure 7.*

PROOF: By a change of the direction in time we can always suppose that $\text{Re}(c_1) < 0$. Then (5.25) has a Hopf bifurcation of order 1 at $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$. The system has an attractive limit cycle for any $\varepsilon_2 \neq 0$ and $\varepsilon_1 > 0$ small enough compared to ε_2 . On this cycle the flow will change direction. The non-generic instance would be a cycle of equilibrium points, but we expect the phenomenon to occur in a more generic way, i.e. through the appearance of a finite number of equilibrium points on the limit cycle. In fact these points appear as q saddle-nodes, which then give $2q$ equilibrium points. These will disappear in q saddle-nodes, but not involving the same pairs of points, such that the total flow direction on the limit cycle has changed.

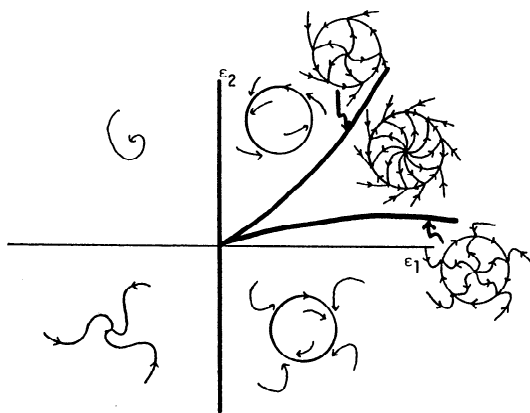


Figure 7. Bifurcation in the case of weak resonance (figure corresponds to resonance of order 5).

To find the singular points explicitly we further reduce (5.25). By change of scale on z and t , we can suppose $A = 1$ and $\text{Re}(c_1) = -1$. Let $c_j = a_j + ib_j$. We bring (5.25) to polar

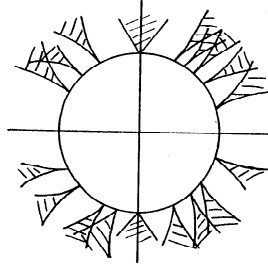


Figure 8. Global bifurcation diagram for eigenvalues in the neighbourhood of the unit circle: "Arnold tongues". Strong resonance effects are not represented.

coordinates

$$\begin{aligned}\dot{r} &= \varepsilon_1 r - r^3 + a_2 r^5 + \dots + r^{q-1} \cos q\theta + O(r^q), \\ \dot{\theta} &= \varepsilon_2 + b_1 r^2 + b_2 r^4 + \dots + r^{q-2} \sin q\theta + O(r^{q-1}).\end{aligned}\quad (5.26)$$

We look for nonzero solutions of $\dot{r} = \dot{\theta} = 0$. (By the implicit function theorem,) the equation $\dot{r}/r = 0$ gives

$$r^2 = \varepsilon_1 + o(\varepsilon) \implies r = \sqrt{\varepsilon_1}(1 + O(\varepsilon)), \quad (5.27)$$

(with $o(\varepsilon)$ and $O(\varepsilon)$ depending on θ). So we have exactly one fixed point for $\dot{r}$, for each fixed θ , and $\varepsilon_1 \geq 0$. We make the following invertible change of coordinates in the second equation

$$\eta_2 = \varepsilon_2 + b_1 r^2 + b_2 r^4 + \dots, \quad (5.28)$$

where r^2 is replaced by its value in equation (5.27). The equation $\dot{\theta} = 0$ becomes

$$\eta_2 - r^{q-2} \sin q\theta + O(r^{q-1}) = 0. \quad (5.29)$$

Under condition (5.27), this equation has solutions in a region approximately of the form

$$|\eta_2| < \varepsilon_1^{(q-2)/2}. \quad (5.30)$$

The boundaries of this cuspidal region correspond to the two curves of simultaneous saddle-node bifurcations. $\square$

REMARK ON THE DIFFERENTIABILITY OF THE INVARIANT CLOSED CURVE. The $2q$ equilibrium points are alternately attractive and repulsive on the cycle, when they exist. Since the cycle itself is attractive, these points are nodes and saddle points. For the saddle points, the two unstable separatrices form a differentiable curve. However, there is no constraint that forces this to happen at the nodes, except on the cuspoid, since the limit cycle without singular points is differentiable. Hence, inside the cuspidal region the limit cycle loses its differentiability. In the case of periodic solutions of vector fields this corresponds to a torus losing its differentiability, and this has been observed numerically.

INTERPRETATION OF THE THEOREM FOR DIFFEOMORPHISMS. The limit cycle for (5.25) corresponds to an attractive closed invariant curve Γ , appearing when the fixed point loses its stability. The cuspidal region corresponds to a pair of orbits of period q on Γ — one attractive, the other a saddle orbit. Outside the cuspidal region, orbits are either non-periodic or periodic of period $q' > q$, (since all rational number p'/q' sufficiently close to p/q satisfy $q' > q$).

We can put all these local results for all weak resonances together in a global picture. This happens if we consider a two-parameter family of diffeomorphisms, where the (real) parameters represent the complex eigenvalue. Then we can draw the different cuspidal regions of the plane (which have attractive periodic orbits) branching from the unit circle (Figure 8). As long as we stay away from the strong resonances, a generic curve in parameter-space will pass through an infinity of cuspidal regions. Hence, after the fixed point has lost its stability, we witness the emergence and disappearance of an infinite number of arbitrarily high-period cycles (no two cycles with different periods can coexist). This phenomenon would have been missed in a one-parameter analysis of loss of stability by a fixed point.

REMARK. Sufficiently close to the unit circle, the region with periodic behavior, (the union of the interiors of the cusps), is open and dense, while approximately of measure zero, while non-periodic behavior is confined to a region of nearly full measure. Accordingly, even if the generic situation is that of periodic motion, in practice (i.e. numerical simulation), we often observe an aperiodic motion as well.

INTERPRETATION OF THE THEOREM IN THE CASE OF BIFURCATION OF PERIODIC SOLUTIONS. Hopf bifurcation in Figure 7 corresponds to the birth of a 2-torus from the periodic solution, while the argument of the eigenvalue on the unit circle corresponds to the rotation number multiplied by 2π . The interior of the cuspoid coincides with the region of rational rotation number $\alpha = p/q$. The flow on the torus thus has at least one periodic solution. More precisely, it has two limit cycles, one attractive and one repulsive, which close after p turns along and q turn around the initial limit cycle. Outside the cuspoid the flow is quasi-periodic. Further to the above remark for diffeomorphisms, we observe that the torus loses its differentiability away from the bifurcation curves.

REMARK. From the point of view of numerical simulation, an attractive torus with an attractive orbit of period q lying on it cannot be distinguished from an attractive orbit of period q alone. Consequently, Figure 8 can be interpreted as the emergence and disappearance of an infinite number of limit cycles of arbitrarily long periods when the initial limit cycle loses its stability.

We now come to the analysis in the neighborhood of strong resonances. In the case $q = 3$ and $q = 4$ we analyze the following systems:

$$\dot{z} = (\varepsilon_1 + i\varepsilon_2)z + \bar{z}^2 + cz^2\bar{z} + O(|z|^4), \quad (5.31)$$

$$\dot{z} = (\varepsilon_1 + i\varepsilon_2)z + \bar{z}^3 + cz^2\bar{z} + O(|z|^4). \quad (5.32)$$

Limiting cases occur where the two eigenvalues approach 1 or -1 . When the two eigenvalues of the linear part of the diffeomorphism are equal, the most generic situation is a linear part with a Jordan block of order 1. Transposed to symmetric vector fields this corresponds to analyzing a vector field whose linear part is a nilpotent Jordan block of order 2, and which has no special symmetry (resp. symmetry under rotation of angle π) in the case of a double eigenvalue equal to 1 (resp. -1). Therefore we study the vector fields

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + ax^2 + bxy + O(|x, y|^3) \end{aligned} \quad (5.33)$$

which we call *strong resonance of order 1* (double eigenvalue 1) and

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 + \varepsilon_2 y + ax^3 + bxy + O(|x, y|^4) \end{aligned} \quad (5.34)$$

which we call *strong resonance of order 2* (double eigenvalue -1). The quadratic (resp. cubic) term in (5.33) (resp. (5.34)) cannot be removed when bringing the system to normal form. In order to have a bifurcation of codimension not greater than 2 we need to take $a, b \neq 0$.

ANALYSIS OF RESONANCE OF ORDER 1 [11]. We now determine the bifurcation diagram of (5.33). A change of scale in x, y, t (allowing time reversal), and in the parameters, brings (5.33) to

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\varepsilon_1 - \varepsilon_2 y + x^2 - xy\end{aligned}\tag{5.35}$$

The singular points of (5.35) are $q_{\pm} = (\pm\sqrt{\varepsilon_1}, 0)$ for $\varepsilon_1 \geq 0$. At $\varepsilon_1 = 0$ two points coincide and form a saddle-node. The point q_+ is a saddle. The point q_- is a focus, attractive for $-\varepsilon_2 + \sqrt{\varepsilon_1} < 0$, and repulsive for $-\varepsilon_2 + \sqrt{\varepsilon_1} > 0$. On the half-parabola, we have a Hopf bifurcation. To determine the type of Hopf bifurcation, we localize the system at the point q_- by the change of variables $X = x + \sqrt{\varepsilon_1}$, and then make the change $y = -\sqrt{\varepsilon_2}Y$. Equation (5.35) becomes

$$\begin{aligned}\dot{X} &= -\sqrt{\varepsilon_2}Y \\ \dot{Y} &= \sqrt{\varepsilon_2}X - \frac{x^2}{\sqrt{\varepsilon_2}} - XY.\end{aligned}\tag{5.36}$$

The origin in (5.36) is a stable singular point (cf. formula in [11] for example). This yields a stable limit cycle in the region $-\varepsilon_2 + \sqrt{\varepsilon_1} > 0$. The cycle soon disappears in a homoclinic loop bifurcation. To approximate this loop we transform the system into a near-Hamiltonian system by means of the change of variables

$$\begin{aligned}x &= \delta^2 u & y &= \delta^3 v & \tau &= \delta t \\ \varepsilon_1 &= \delta^4 & \varepsilon_2 &= \delta \mu_2\end{aligned}\tag{5.37}$$

The new system is (where ' denotes the derivative with respect to τ)

$$\begin{aligned}u' &= v \\ v' &= -1 - \mu_2 v + u^2 + \delta uv.\end{aligned}\tag{5.38}$$

For $\mu_2 = \delta = 0$, we have a Hamiltonian system with Hamiltonian function

$$H(u, v) = \frac{v^2}{2} - \frac{u^3}{3} + u.\tag{5.39}$$

The level curve $H(u, v) = 2/3$ is a homoclinic loop γ for the Hamiltonian system (Figure 9a). The homoclinic loop of system (5.38) can be approximated by equation [11]

$$M(\mu_2, \delta) = \int_{\gamma} -\mu_2 v - \delta uv \, du = 0.\tag{5.40}$$

i.e. the line integral of the perturbation is zero. Substituting $v = \pm\sqrt{2/3}(1-u)\sqrt{u+2}$ in (5.40) we get $\mu_2/\delta = 5/7$. From (5.37) we get $\varepsilon_2 = 5/7\sqrt{\varepsilon_1}$. The bifurcation diagram is given in Figure 10.

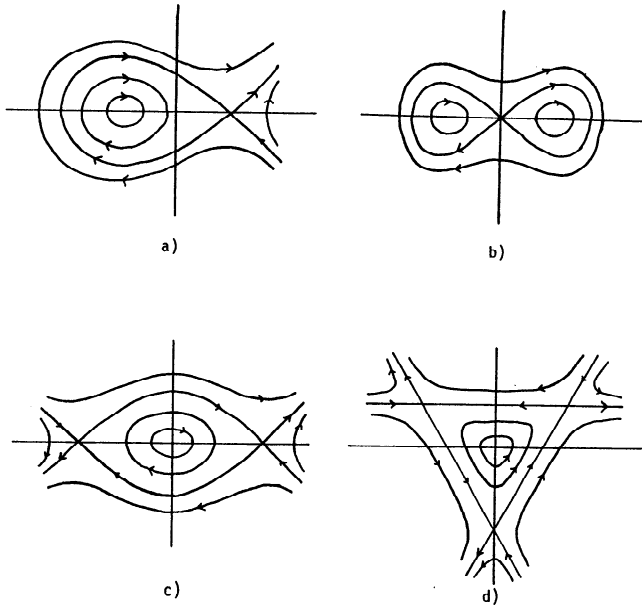


Figure 9. Phase portraits of Hamiltonian systems with Hamilton functions: a) (5.39), b) (5.45), c) (5.51), d) (5.59).

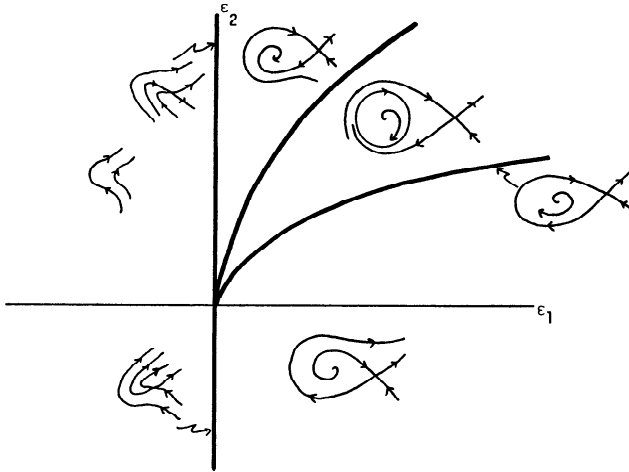


Figure 10. Bifurcation diagram in the case of strong resonance of order 1.

INTERPRETATION OF RESONANCE OF ORDER 1.

Map. In this case the emergence of $q_{\pm}$ at $\varepsilon_1 = 0$ corresponds to the emergence of two fixed points of the map. Hopf bifurcation corresponds to the appearance of an attractive invariant closed curve for the map. This attractive closed curve then interacts with the stable and unstable manifolds of the saddle point. However, in the generic case of a map, the stable and unstable manifolds of a saddle point can cut transversally. (Remember that (5.33) is only a mean value for the case of resonance 1, so that long-term features appear in the dynamics of (5.33), but some details are not present.) In this case, they cross in an infinite number of points (Figure 11a). This results in

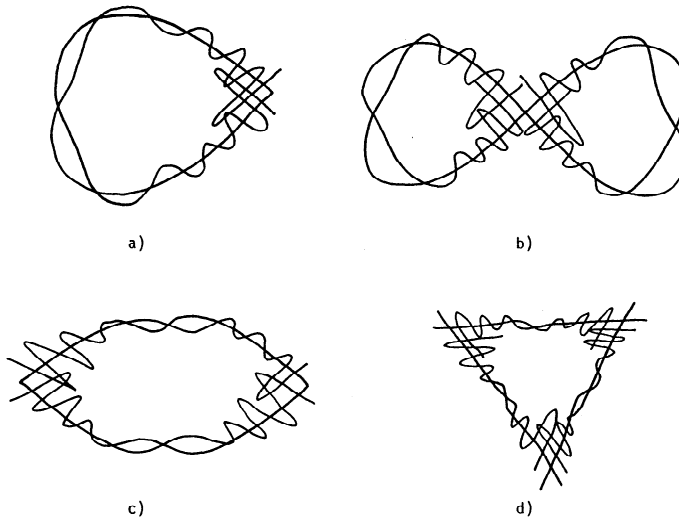


Figure 11. Transversal intersection of stable and unstable manifolds of periodic points for the resonances a) of order 1, b) and c) of order 2, d) of order 3.

an infinite number of horseshoes and chaotic behavior in the dynamics (*cf.* [11]).

Flow. We have the same interpretation with the two fixed points replaced by a pair of periodic orbits, and the invariant closed curve by an invariant torus. The homoclinic orbit corresponds to the destruction of the torus, when it intersects the stable and unstable manifolds of the other periodic orbit. Generically the two surfaces can cut transversally (Figure 12). The intersection curve is not closed, but winds around an infinite number times. The flow is everywhere tangent to the curve.

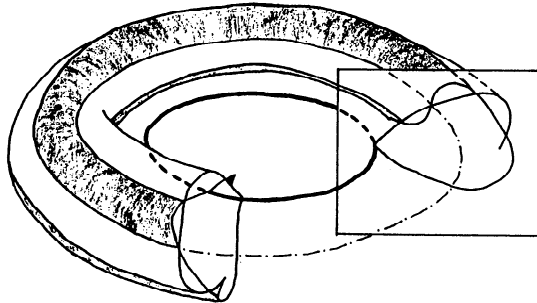


Figure 12. Transversal intersection of stable and unstable manifolds of periodic solution in the case of a homoclinic bifurcation for resonance of order 1.

ANALYSIS OF RESONANCE OF ORDER 2. We analyze system (5.34). Rescaling and neglecting higher order terms brings it to

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \epsilon_1 x + \epsilon_2 y \pm x^3 - x^2 y.\end{aligned}\tag{5.41}$$

The two cases with $\pm$ signs are qualitatively different. We start with the minus sign.

Bifurcation diagram of (Figure 13)

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \varepsilon_1 x + \varepsilon_2 y - x^3 - x^2 y.\end{aligned}\tag{5.42}$$

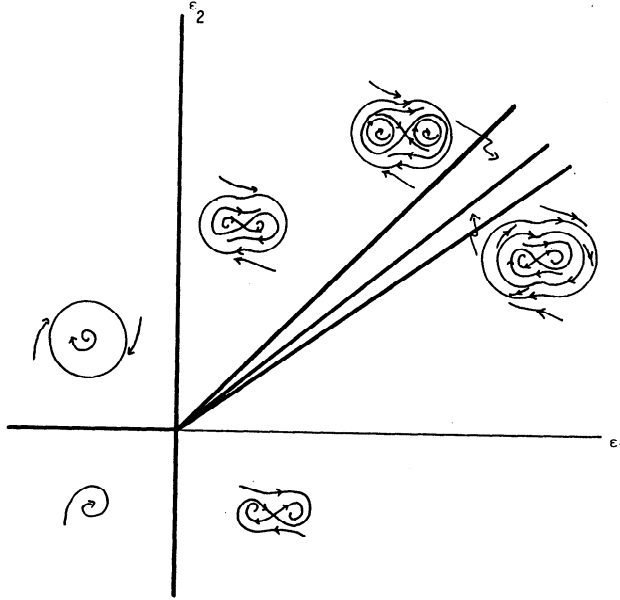


Figure 13. Bifurcation diagram for resonance of order 2: system (5.42).

- i) The system has one singular point, the origin, for $\varepsilon_1 < 0$. The origin is attractive for $\varepsilon_2 < 0$ and repulsive for $\varepsilon_2 > 0$. It has a Hopf bifurcation at $\varepsilon_2 = 0$, and an attractive limit cycle for $\varepsilon_2 > 0$.
- ii) At $\varepsilon_1 = 0$ the system has a pitchfork bifurcation. The origin becomes a saddle point, and two singular points $q_{\pm} = (\sqrt{\varepsilon_1}, 0)$ appear. The points $q_{\pm}$ are attractive for $\varepsilon_2 < \varepsilon_1$, and repulsive for $\varepsilon_2 > \varepsilon_1$, with a Hopf bifurcation on the line $\varepsilon_2 = \varepsilon_1$. They are both surrounded by a repulsive limit cycle for $\varepsilon_2 < \varepsilon_1$.
- iii) The divergence of the vector field $\text{div} = \varepsilon_2 - x^2$ is non-zero for $\varepsilon_2 < 0$. Hence all limit cycles must disappear when we reach the region $\varepsilon_1 > 0, \varepsilon_2 < 0$. We therefore look for homoclinic loops, making the change of variables

$$\begin{aligned}x &= \delta u & y &= \delta^2 v & \tau &= \delta t \\ \varepsilon_1 &= \delta^2 & \varepsilon_2 &= \delta \mu_2\end{aligned}\tag{5.43}$$

The system becomes

$$\begin{aligned}u' &= v \\ v' &= u + \mu_2 v - u^3 - \delta u^2 v.\end{aligned}\tag{5.44}$$

For $\mu_2 = \delta = 0$, we have a Hamiltonian system, with Hamiltonian function

$$H(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4}.\tag{5.45}$$

The Hamiltonian system has a double homoclinic loop γ , given by $H = 0$ (Figure 9b). An approximation of the double homoclinic loop of system (5.42) is given by

$$M(\mu_2, \delta) = \int_{\gamma} (\mu_2 v - \delta u^2 v) du = 0. \quad (5.46)$$

Since $v = u/2(1 - u^2/2)^{1/2}$ we get

$$\mu_2 = \frac{4}{5}\delta \text{ i.e. } \varepsilon_2 = \frac{4}{5}\varepsilon_1 \quad (5.47)$$

Before the loop occurs the "object" formed by the three singular points and the separatrices of the origin is repulsive. However the trace of the origin is positive, so the loop makes repulsive cycles appear or disappear (cf. [1]). We conclude that the loop makes the two repulsive cycles around q_+ disappear. After the loop the "object" becomes attractive. So we need a repulsive limit cycle around the "object". Since there are no other homoclinic loops, the only way the two cycles disappear before the region $\varepsilon_2 < 0$, is if they coalesce (as in Figure 1b).

INTERPRETATION OF BIFURCATION DIAGRAM OF (5.42) FOR FIXED POINTS WITH A DOUBLE -1 EIGENVALUE.

Map. Suppose we start in the region $\varepsilon_1, \varepsilon_2 < 0$, where the origin is a stable fixed point. The fixed point can lose its stability in two ways. The first is when $\varepsilon_1 = 0$: one eigenvalue becomes equal to -1 , yielding a stable orbit of period 2. The second is when $\varepsilon_2 = 0$: the two eigenvalues cross the unit circle and we have an attractive invariant closed curve C_1 . The initial fixed point is now repulsive. When we cross the semi-axis $\varepsilon_1 = 0, \varepsilon_2 > 0$, one eigenvalue becomes equal to -1 , creating an unstable orbit of period 2 inside C_1 . The eigenvalues of the unstable period 2 orbit cross the unit circle, and consequently we get a pair of repulsive closed curves around the period 2 orbit: the set formed by these two curves is globally invariant, and the iterates alternate between the two. These invariant closed curves grow until they interact with the stable and unstable manifolds of the origin. The unstable invariant closed curves expand, and, through a chaotic process involving transversal crossings of the stable and stable manifolds (Figure 11b), they amalgamate in a large repulsive invariant closed curve C containing the origin, its separatrices and the two points of period 2. The two curves C_1 and C_2 then coalesce and disappear.

Flow. We have a similar interpretation with the fixed point replaced by a periodic orbit, the periodic points replaced by a periodic orbit of period 2, and the invariant closed curves by invariant tori. The two limit cycles of (5.24) around $q_{\pm}$ correspond to a unique torus twisted twice, centered on a periodic orbit of period 2 (Figure 25).

REMARK. The behavior is more complex than in the case of weak resonances, since the periodic points need not appear on an invariant closed curve. Accordingly, there is some interaction between the different periodic points and the invariant closed curve.

We now come to the second case of (5.41), and study

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 x + \varepsilon_2 y + x^3 - x^2 y. \end{aligned} \quad (5.48)$$

BIFURCATION DIAGRAM OF (5.48) (FIGURE 14).

- i) The origin is a saddle point for $\varepsilon_1 > 0$, and there are no other singular points.

- ii) For $\varepsilon_1 = 0$ we have a pitchfork bifurcation, and two saddle points $q_{\pm} = (\pm\sqrt{-\varepsilon_1}, 0)$ appear for $\varepsilon_1 < 0$. The origin is attractive for $\varepsilon_2 < 0$ and repulsive for $\varepsilon_2 > 0$. A limit cycle appears for $\varepsilon_2 > 0$ via Hopf bifurcation.
- iii) $\text{div} = \varepsilon_2 - x^2$, so the limit cycle must disappear for $\varepsilon_2 > \varepsilon_1$ (via heteroclinic connection between $q_{\pm}$). The change of variables

$$\begin{aligned} x &= \delta u & y &= \delta v & \tau &= \delta t \\ \varepsilon_1 &= -\delta^2 & \varepsilon_2 &= \delta\mu_2, \end{aligned} \quad (5.49)$$

brings (5.48) to

$$\begin{aligned} u' &= v \\ v' &= -u + \mu_2 v + u^3 - \delta u^2 v. \end{aligned} \quad (5.50)$$

which is a perturbation of a Hamiltonian system with Hamiltonian function

$$H(u, v) = \frac{v^2}{2} + \frac{u^2}{2} - \frac{u^4}{4}. \quad (5.51)$$

The heteroclinic connection γ at $H = 1/4$ (Figure 9c), stays for

$$M(\mu_2, \delta) = \int_{\gamma} (\mu_2 v - \delta u^2 v) du = 0 \text{ i.e. } \delta = 5\mu_2 \text{ i.e. } \varepsilon_2 = -\frac{\varepsilon_1}{5}. \quad (5.52)$$

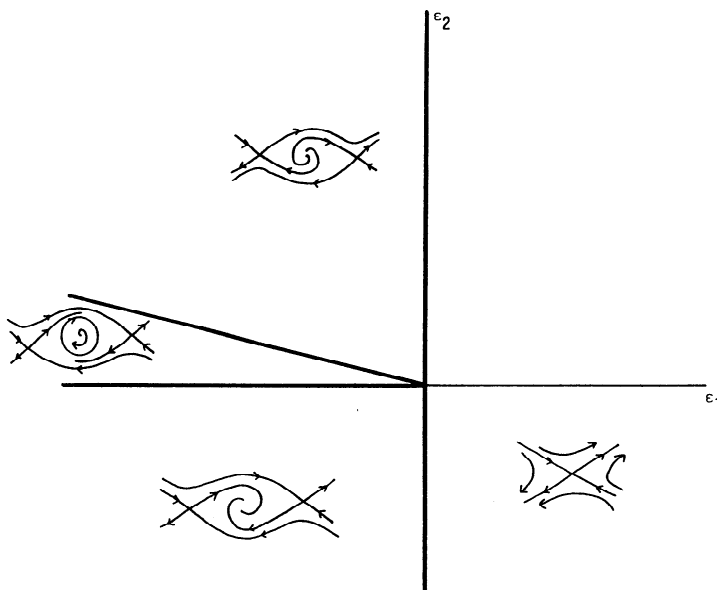


Figure 14. Bifurcation diagram for resonance of order 2: system (5.48).

INTERPRETATION OF BIFURCATION DIAGRAM OF (5.48).

Map. In the region $\varepsilon_1 > 0$ the origin is a saddle point. It becomes attractive (repulsive) when it crosses the semi-axis $\varepsilon_2 < 0$ ($\varepsilon_2 > 0$), where a saddle orbit of period 2

appears. The origin loses its stability on the semi-axis $\varepsilon_1 < 0$, and an invariant attractive closed curve appears for $\varepsilon_2 > 0$. This invariant closed curve disappears in a heteroclinic connection for the orbit of period 2: the stable manifold of each periodic point intersects the unstable manifold of the other periodic point transversally (figure 11c).

Flow. Same interpretation, with the origin replaced by a periodic orbit, the points $q_{\pm}$ replaced by a periodic orbit of period 2, and the invariant closed curve by an attractive torus.

REMARK. The difference between (5.42) and (5.48) lies essentially in the direction of the period doubling bifurcation. In the first case the fixed point loses its stability, yielding a stable orbit of period 2. In the second case there are no regions of the $\varepsilon_1 - \varepsilon_2$ plane where we have only one stable fixed point. The stable fixed point collapses with a saddle orbit of period 2 on the $\varepsilon_2 < 0$ semi-axis to give a fixed saddle point.

ANALYSIS OF THE RESONANCE OF ORDER 3 [2]. Here, we analyze the phase portrait of

$$\dot{z} = (\varepsilon_1 + i\varepsilon_2)z + c_1 z^2 \bar{z} + \bar{z}^2. \quad (5.53)$$

A change of variables

$$\begin{aligned} z &= \delta Z & \varepsilon_1 &= \delta \mu_1 \\ \varepsilon_2 &= \delta \mu_2 & \tau &= \delta t \end{aligned} \quad (5.54)$$

brings (5.53) to the form

$$\dot{Z} = (\mu_1 + i\mu_2)Z + c_1 \delta Z^2 \bar{Z} + \bar{Z}^2. \quad (5.55)$$

This system is Hamiltonian for $\mu_1 = \delta = 0$. To analyze the phase portrait of (5.53) we use (5.53) or (5.55).

BIFURCATION DIAGRAM OF (5.53) (FIGURE 15). What makes this case different is that for any $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$ the system (5.53) has always three singular points in the neighborhood of the origin. In fact, singular points for (5.55) at $\delta = 0$ are given by $Z = re^{i\theta}$ satisfying

$$re^{i\theta}[(\mu_1 + i\mu_2) + re^{-3i\theta}] = 0. \quad (5.56)$$

This has the following solutions, corresponding to three singular points

$$\mu_1^2 + \mu_2^2 = r^2, \quad \theta = -\frac{\phi}{3} + 2\pi \frac{k}{3} \quad \text{where } \mu = \mu_1 + i\mu_2 = re^{i\phi}. \quad (5.57)$$

The Jacobian matrix is given by

$$A = \begin{pmatrix} \mu & 2\bar{z} \\ 2z & \bar{\mu} \end{pmatrix}. \quad (5.58)$$

The determinant $|A| = |\mu|^2 - 4|z|^2$ is negative at the singular points, so we have three saddle points. Since saddle points are structurally stable, this property also holds for (5.53).

- ii) The origin in (5.53) is attractive for $\varepsilon_1 < 0$ and repulsive for $\varepsilon_1 > 0$. It has a Hopf bifurcation at the origin for $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$. This gives an attractive limit cycle for sufficiently small $\varepsilon_1 > 0$ (if we suppose $\text{Re}(c_1) < 0$).

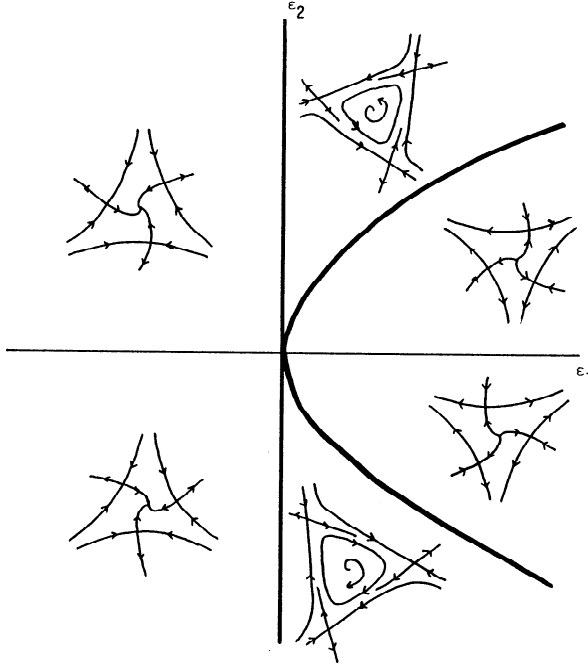


Figure 15. Bifurcation diagram for resonance of order 3.

- iii) The divergence of the vector field is $\text{div} = 2\varepsilon_1 + 4\text{Re}(c_1)(x^2 + y^2) > 0$ in the region $\varepsilon_1 > 2|\text{Re}(c_1)|(\varepsilon_1^2 + \varepsilon_2^2)$, in particular on the semi-axis $\varepsilon_1 > 0$ close to the origin. The limit cycle around the origin disappears in a heteroclinic connection joining the three saddle points. The curve in parameter-space for which we have the connection can be approximated using the similar connection γ (Figure 9d) for the Hamiltonian system

$$\dot{Z} = i\mu_2 Z + \bar{Z}^2. \quad (5.59)$$

Here, as before, we approximate the curve by integrating the perturbation over γ , (Figure 9d).

$$\begin{aligned} M(\mu_1, \delta) &= \int_{\gamma} (\mu_1 Z + \delta c_1 Z^2 \bar{Z}) d\bar{Z} + (\mu_1 \bar{Z} + \delta c_1 Z \bar{Z}^2) dZ \\ &= 3 \int_{-\mu_2 \sqrt{3}/2}^{\mu_2 \sqrt{3}/2} [\mu_1 y + \delta \text{Re}(c_1) y(x^2 + y^2) + \delta \text{Im}(c_1) x(x^2 + y^2)] \Big|_{y=\mu_2/2} dx \\ &= \mu_2^2 \frac{\sqrt{3}}{2} \left[\mu_1 + \delta \text{Re}(c_1) \frac{\mu_2^2}{4} \right] = 0 \text{ i.e. } \mu_1 = -\delta \text{Re}(c_1) \frac{\mu_2^2}{4} \end{aligned} \quad (5.60)$$

This yields an approximation by the parabola $\varepsilon_1 = -\text{Re}(c_1)\varepsilon_2^2/4$ for system (5.53).

INTERPRETATION OF THE BIFURCATION DIAGRAM OF (5.53).

Map. We note that for any $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$ we have a periodic orbit of order 3, which is of saddle type. This is better understood if we look at the behavior of the map at $(\varepsilon_1, \varepsilon_2) = (0, 0)$ (Figure 21 thereafter and ref. [15]). The fixed point is stable for $\varepsilon_1 < 0$, and undergoes a Hopf bifurcation for $\varepsilon_1 = 0$. For $\varepsilon_1 > 0$ sufficiently small

compared to ε_2 , we have an attractive invariant closed curve. The closed curve disappears during transversal intersection of the stable and unstable manifolds of period-three orbit (Figure 11d).

Flow. As the case of a map, we always have a saddle-type orbit of period 3. The invariant closed curve corresponds to a torus, which disappears when it intersects the stable and unstable manifolds of the orbit of period 3.

REMARK ON RESONANCES OF ORDER 1, 2 AND 3. In these cases it is shown that (5.35), (5.42), (5.48), (5.53) are universal unfoldings of the corresponding systems at $\varepsilon_1 = \varepsilon_2 = 0$ ([3], [4], [12]), and that the addition of higher order terms does not change the universal unfoldings.

ABOUT RESONANCE OF ORDER 4. In this case we study the phase portrait of

$$\dot{z} = \varepsilon z + c_1 z^2 \bar{z} + \bar{z}^3, \quad \varepsilon = \varepsilon_1 + i\varepsilon_2. \quad (5.61)$$

This study is incomplete. It is almost complete for the case where the system (5.61) can be brought to a near-Hamiltonian system through changes of coordinates similar to (5.54). We will sketch the known results. In the case $|c_1| < 1$, (5.61) always has four singular points for $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$, and this yields a bifurcation diagram similar to Figure 15, with a similar interpretation (orbit of period 3 replaced by orbit of period 4). In the case $|\operatorname{Re}(c_1)| > 1$ we get a bifurcation diagram similar to the case of weak resonances. The difficulties arise when $|c_1| > 1$ $|\operatorname{Re}(c_1)| < 1$.

Singular points of (5.61). Suppose that $\operatorname{Re}(c_1), \operatorname{Im}(c_1) < 0$. In polar coordinates $z = re^{i\theta}$, the equation $\dot{z} = 0$ gives

$$-\frac{\varepsilon}{r^2} = c_1 + e^{-4i\theta}. \quad (5.62)$$

The point $c_1 + e^{-4i\theta}$ lies on the circle C centered at c_1 with radius 1. There are two different cases, depending on whether the origin is contained within C or not (Figure 16).

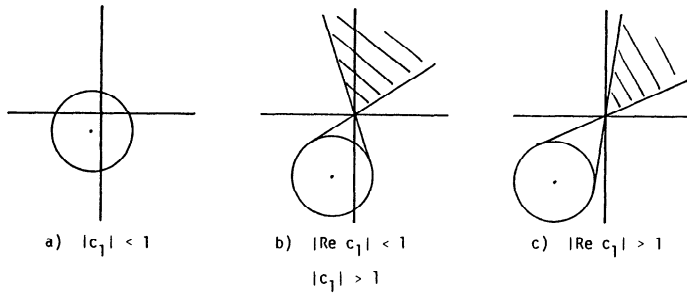


Figure 16. Possible positions of C . The dark region in b) and c) indicates the region of the ε -plane where there are four singular points distinct from the origin.

- i) $|c_1| < 1$. In this case the origin lies inside C . To solve (5.62) we must choose θ such that ε and $c_1 + e^{-4i\theta}$ on C have opposite directions. This is always possible (Figure 16a). Thus, for any $\varepsilon \neq 0$, (5.61) always has four fixed points.
- ii) $|c_1| > 1$. Now ε must be chosen with direction opposite to any vector joining the origin to a point of C . This is possible only for ε in an angular sector bounded by the semi-lines opposite to the tangent lines to C , issuing from the origin (Figure 16b and 16c). For such an ε , there are two points of the circle with direction opposite to ε , so we get eight fixed points for the system.

Stability of the fixed points. The Jacobian matrix at a fixed point $z_0 = re^{i\theta}$ is

$$A = \begin{pmatrix} \varepsilon + 2c_1 r^2 & r^2 e^{2i\theta} (c_1 + 3e^{-4i\theta}) \\ r^2 e^{-2i\theta} (\bar{c}_1 + 3e^{4i\theta}) & \bar{\varepsilon} + 2\bar{c}_1 r^2 \end{pmatrix}. \quad (5.63)$$

We have a saddle if $|A| < 0$, i.e.

$$|\varepsilon + 2c_1 r^2| < |r^2 e^{2i\theta} (c_1 + 3e^{-4i\theta})|. \quad (5.64)$$

Replacing ε by $-r^2(c_1 + e^{-4i\theta})$ using (5.62), (5.64) is equivalent to

$$|c_1 - e^{-4i\theta}| < |c_1 + 3e^{-4i\theta}|. \quad (5.65)$$

- i) If $|c_1| < 1$, the left member is less than 2, and the right member is greater than 2, so the four fixed points are always saddle points.
- ii) If $|c_1| > 1$, the eight singular points appear as diametrically opposed pairs on C . In this case, one satisfies $|A| < 0$ (i.e. it is a saddle) and the other satisfies $|A| > 0$ (i.e. it is a node or a focus). Its stability is then determined by $\text{Tr}(A)$. Since

$$\text{Tr}(A) = 2 \text{Re}(\varepsilon + 2c_1 r^2) = 2r^2 \text{Re}(c_1 - e^{-4i\theta}), \quad (5.66)$$

for $\text{Re}(c_1) < -1$, $\text{Tr}(A)$ has constant sign: the sign of $\text{Re}(c_1)$. For $\text{Re}(c_1) > -1$, $\text{Tr}(A)$ may take different values, when ε varies. In this case we may have Hopf bifurcations at four of the singular points. The saddle corresponds to the points on C which is closest to the origin, and so the saddles are closer to the origin than the nodes or foci.

Hopf Bifurcations. The origin has a Hopf bifurcation at $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$, which yields an attractive limit cycle for $\varepsilon_1 > 0$. We calculate the condition for which the four foci can have a Hopf bifurcation in the case $|c_1| > 1$, $|\text{Re}(c_1)| < 1$. If we define $N = e^{-4i\theta}$, Hopf bifurcation for the points occurs when the point $c_1 - N$ lies on the imaginary axis. But this point $c_1 - N$ must correspond to a focus and not to a node, so the point $c_1 + N$ must be the point of C closest to the origin and lying on the line from the origin to $c_1 + N$. The boundary of the region where we have Hopf bifurcation is the case where $c_1 - N$ is on the imaginary axis, and the line joining $c_1 + N$ to the origin is tangent at C (Figure 17a). It is not difficult to see that the corresponding condition is

$$|\text{Im}(c_1)| = \frac{1 + \text{Re}(c_1)^2}{\sqrt{1 - \text{Re}(c_1)^2}}. \quad (5.67)$$

The corresponding curve in the space $\text{Re}(c_1)$, $\text{Im}(c_1)$ is illustrated in Figure 17b together with the region where Hopf bifurcation occurs.

Tentative bifurcation diagrams for (5.61). We need to suppose that (5.61) satisfies $|c_1| \neq 1$, $|\text{Re}(c_1)| \neq 1$, $|\text{Im}(c_1)| \neq (1 + \text{Re}(c_1)^2)/\sqrt{1 - \text{Re}(c_1)^2}$, otherwise we have a bifurcation of order greater than two. There are probably other non-degeneracy conditions, but since the study is not finished, they are still unknown.

- i) $|c_1| < 1$. In this case the bifurcation diagram is probably the same as in resonance of order 3 (Figure 18).

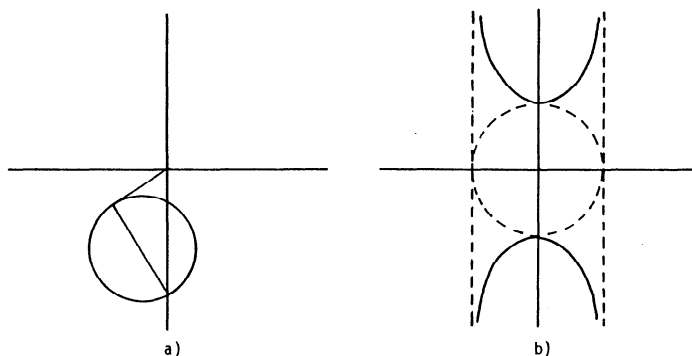


Figure 17. a) Limiting position of C in the ε -plane, for appearance of Hopf bifurcation, b) Degeneracy curves for the resonance of order 4, separating regions of the c_1 -plane corresponding to different bifurcation diagrams (probably not an exhaustive list).

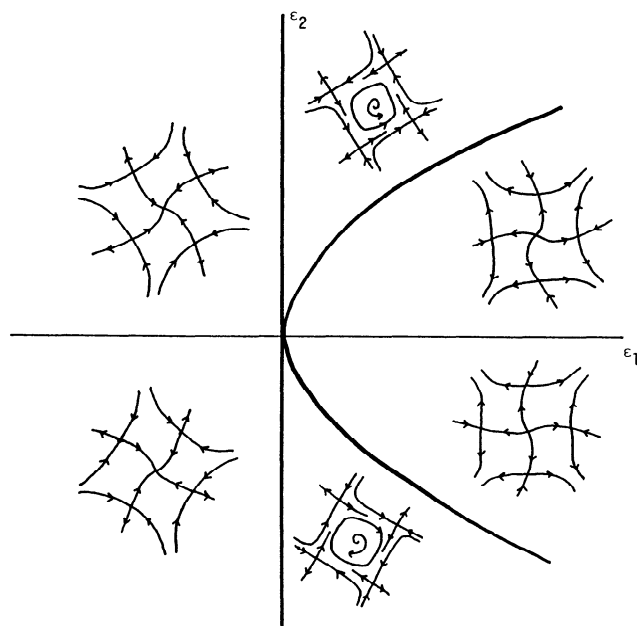


Figure 18. Bifurcation diagram for resonance of order 4, case $|c_1| < 1$.

- ii) $|\operatorname{Re}(c_1)| > 1$. The bifurcation diagram is probably the same as in the case of weak resonances, the only difference being that the region with eight singular points is bounded by two straight lines with a non-zero angle at the origin (Figure 19).
- iii) $|\operatorname{Re}(c_1)| < 1$, $|c_1| > 1$. Here, the only possible Hopf bifurcation are necessarily of order 1 [17]. The saddle node bifurcations happen in two half-lines, one on the left of the ε_2 -axis, the other on the right. So for Hopf bifurcation at $\varepsilon_1 = 0$, $\varepsilon_2 < 0$ (resp. $\varepsilon_2 > 0$) the eight singular points lie inside (resp. outside) the attractive limit cycle. Arnold [2] gives a tentative bifurcation diagram for the case $\operatorname{Im}(c_1) \rightarrow \infty$ (Figure 20). This can be scaled to a system

$$\dot{z} = \varepsilon z + (\alpha + i)z^2\bar{z} + \beta\bar{z}^3, \quad (5.67)$$

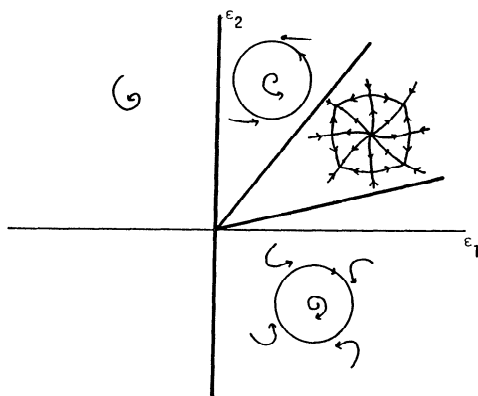


Figure 19. Bifurcation diagram for resonance of order 4, case $|\operatorname{Re}(c_1)| > 1$.

with α, β small. The heteroclinic bifurcations can be studied as in the previous cases, using the heteroclinic connections of the Hamiltonian system corresponding to (5.67) with $\varepsilon_1 = \alpha = 0$. We refer the reader to [2] for the details. (Note that the precise direction of bifurcation curves at the origin has not been calculated, and the reader should only consider the topological form of the bifurcation diagram.)

REMARKS.

- (1) We can explain the different shapes of the bifurcation diagrams by the phase portrait of the system at $\varepsilon = 0$ (cf. [15] and Figure 21).
 - i) resonance of order 3 and order 4 (for $|c_1| < 1$).
 - ii) weak resonance and resonance of order 4 (for $|\operatorname{Re}(c_1)| > 1$), and first case of order 2 (corresponding to (5.42)).
 - iii) resonance of order 1.
 - iv) second case of order 2 (corresponding to (5.48)).
 - v) resonance of order 4 (for $|c_1| > 1$ and $|\operatorname{Re}(c_1)| < 1$).

The effect of the unfolding can be viewed as “blowing up the origin”. In case i) there are always trajectories going to and from the origin, before perturbation. The effect of the perturbation is to transform the origin into an “object” with the same property. Similarly, the origin is attractive in ii) and v) for $\varepsilon = 0$ and the perturbation blows it up into an “attractive object”. The same remark applies for the other cases.

- (2) Our whole analysis has been for $\operatorname{Re}(c_1) < 0$. The case where $\operatorname{Re}(c_1) > 0$ is similar, except that all directions of the vector field are reversed.

REMARK. In all resonances we have neglected to prove the versality of the bifurcation diagrams described. The proofs exist, except for the resonance of order 4. Since they are rather technical and of no interest for the applications, we have omitted them. The simplest reference for the resonance of order 1 is [9], although the original proof was due to Bogdanov [3]. For the resonance of order 2 we refer to [4], [12], and for the case of order 3 to [8] or [12].

6. Bifurcation with eigenvalues 1 and -1 . To complete the study of bifurcations of codimension 2, we have evidently the case with two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. In

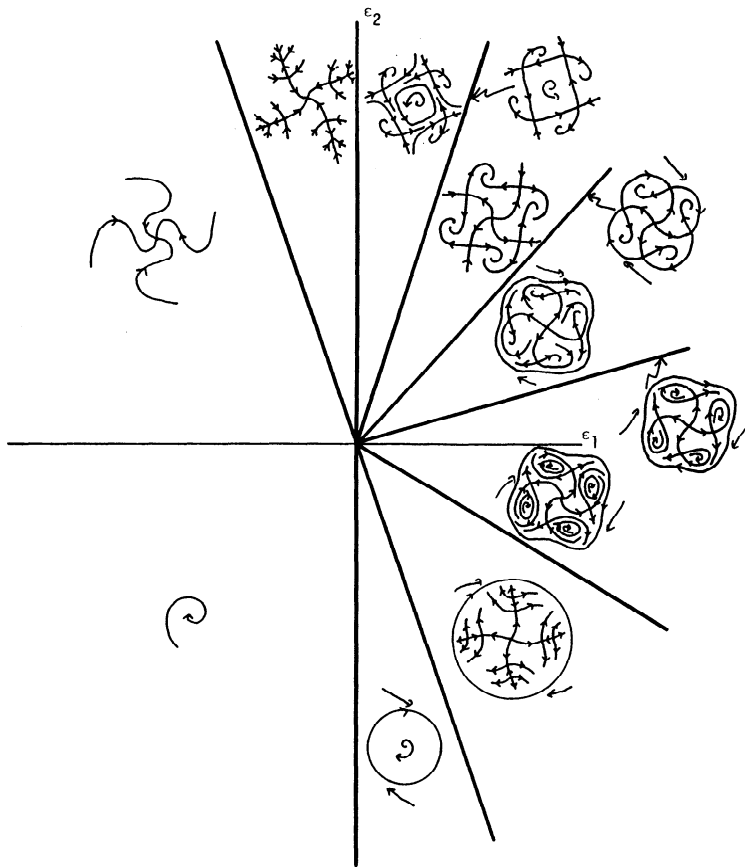


Figure 20. Tentative bifurcation diagram for resonance of order 4, case $\text{Re}(c_1) < 1$, $\text{Im}(c_1) \rightarrow \infty$.

this case we have the resonances (2.12), which give the map (2.13).

Here, as in section 5, instead of directly studying (2.13), we study symmetric vector fields. We start with a vector field which has a periodic solution, corresponding to a fixed point of the Poincaré map on a cross-section, with eigenvalues ± 1 .

- i) We localize the vector field at the periodic solution, and apply Floquet theory to get

$$\begin{aligned} \dot{x} &= f(x, y, t) \quad f(x, y, t) = O(|x, y|^2) \\ \dot{y} &= \frac{iy}{2} + g(x, y, t) \quad f(x, y, t) = O(|x, y|^2) \end{aligned} \quad (6.1)$$

with f and g periodic in t of period 2π . We can take

$$\begin{aligned} f &= \sum a_{m,k} x^{m_1} y^{m_2} e^{ikt} = f_r(x, y, t) + O(|x, y|^{r+1}), \\ g &= \sum b_{m,k} x^{m_1} y^{m_2} e^{ikt} = g_r(x, y, t) + O(|x, y|^{r+1}). \end{aligned} \quad (6.2)$$

- ii) As in case of normal form, we perform a change of variables

$$\begin{aligned} x &= X + \sum c_{m,k} X^{m_1} Y^{m_2} e^{ikt} = X + h_r(X, Y, t) + O(|X, Y|^{r+1}), \\ y &= Y + \sum d_{m,k} X^{m_1} Y^{m_2} e^{ikt} = Y + k_r(X, Y, t) + O(|X, Y|^{r+1}). \end{aligned} \quad (6.3)$$

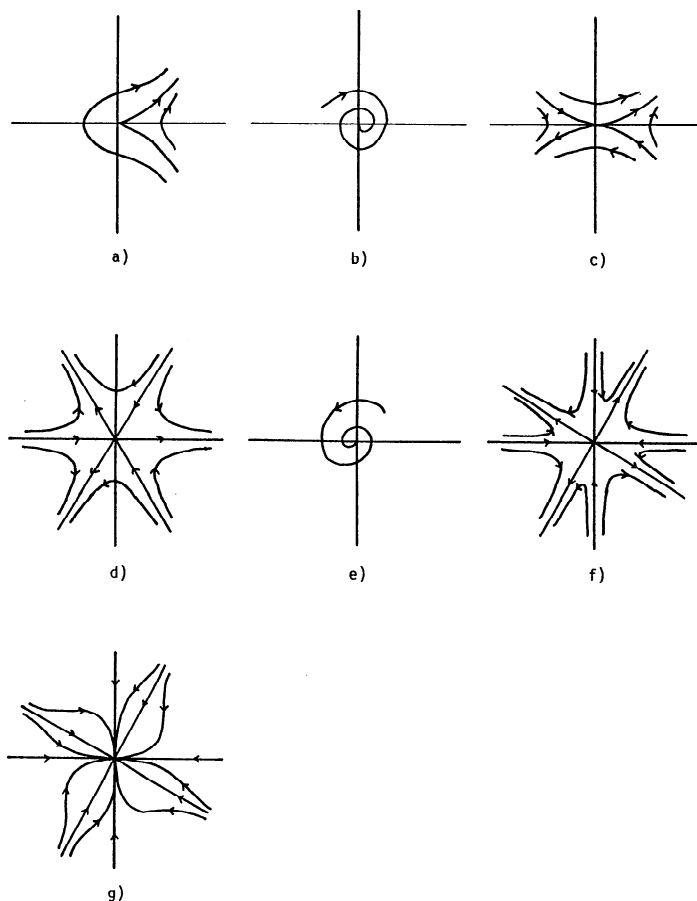


Figure 21. Configuration of resonances before the unfoldings: a) order 1, b) order 2 with (5.41), c) order 2 with (5.47), d) order 3 e) order 4 in case $\text{Re}(c_1) > 1$ and weak resonance, f) order 4 in case $|c_1| < 1$, g) order 4 in case $\text{Re}(c_1) < 1$ and $|c_1| > 1$.

in order to eliminate as much terms as possible in (6.1), and solve the homological equation [2], as in Section 5:

$$\begin{aligned} h_r &= \frac{iY}{2} \frac{\partial h_r}{\partial Y} + \frac{\partial h_r}{\partial t}, \\ k_r &= \frac{iY}{2} \frac{\partial k_r}{\partial Y} + \frac{\partial k_r}{\partial t} - \frac{i\pi}{2} k_r. \end{aligned} \quad (6.4)$$

An h_r (resp. k_r) of the form $x^{m_1} y^{m_2} e^{ikt}$ is an eigenvector with eigenvalue $\lambda = im_2/2 + ik$ (resp. $\lambda = i(m_2 - 1)/2 + ik$). Consequently, the normal form is

$$\begin{aligned} \dot{X} &= \sum C_{m,k} X^m Y^{2k} e^{-ikt}, \\ \dot{Y} &= \sum D_{m,k} X^m Y^{2k+1} e^{-ikt}. \end{aligned} \quad (6.5)$$

Here X and Y are complex variables.

iii) We make the change of coordinates $Y = e^{it/2}Z$. Then (6.5) becomes

$$\begin{aligned}\dot{X} &= \sum C_{m,k} X^m Z^{2k}, \\ \dot{Z} &= \sum D_{m,k} X^m Z^{2k+1},\end{aligned}\tag{6.6}$$

which is now a real normal form.

Now, to reduce the study of (2.13) to (6.6), we can construct (as in Section 5) a symmetric vector field in $\mathbb{R}^2 \times S^1$, with flow map at time 2π equal to F^2 , to arbitrarily large order

$$\begin{aligned}\dot{x} &= \sum a_m x^{m_1} y^{2m_2} + (O(|x, y|^{r+1})), \\ \dot{y} &= \sum b_m x^{m_1} y^{2m_2+1} + (O(|x, y|^{r+1})), \\ \dot{t} &= 1.\end{aligned}\tag{6.7}$$

A candidate for the universal unfolding of (2.13) is the map

$$F_\varepsilon(x, y) = (\varepsilon_1 + x + ax^2 + by^2 + Ax^3 + Bxy^2 + \dots, -(1 + \varepsilon_2)y + cxy + Cx^2y + Dy^3),\tag{6.8}$$

which is studied by means of the vector field

$$\begin{aligned}\dot{x} &= \varepsilon_1 + ax^2 + by^2 + Ax^3 + Bxy^2, \\ \dot{y} &= \varepsilon_2 y + cxy + Cx^2y + Dy^3.\end{aligned}\tag{6.9}$$

This field is symmetric about the x -axis.

STUDY OF THE VECTOR FIELD (6.9). By rescaling on x , y , and making a translation on x , we can study the system [19]

$$\begin{aligned}\dot{x} &= \varepsilon_1 + \varepsilon_2 x + ax^2 + by^2 + Ax^3 + Bxy^2, \\ \dot{y} &= -2xy + Cx^2y + Dy^3, \quad b = \pm 1.\end{aligned}\tag{6.10}$$

A change of coordinates

$$\begin{aligned}x &= X + \alpha X^2 + \beta XY, \\ y &= Y + \gamma XY, \\ \tau &= t(1 + \delta X),\end{aligned}\tag{6.11}$$

reduces the study of (6.10) to the study of (6.12):

$$\begin{aligned}\dot{x} &= \varepsilon_1 + \varepsilon_2 x + ax^2 + by^2 + cx^3, \\ \dot{y} &= -2xy, \quad b = \pm 1,\end{aligned}\tag{6.12}$$

and $a \neq 0$, $c = 0$ if $b = -1$, $a \neq 0, -2$, $c = \pm 1$ if $b = 1$. We have four different, non-equivalent cases for the phase portraits of (6.12), depending on the signs of a and b .

SINGULAR POINTS OF (6.12). They lie on the coordinate-axis. Since the x -axis is invariant, there is no complex behavior around the singular points on the x -axis.

- i) The singular points on the y -axis are $p_\pm = (0, \pm\sqrt{|\varepsilon_1/b|})$ for $\varepsilon_1 b < 0$. The determinant of the linearized vector field is $4by^2$, so we have two saddles if $b = -1$, and two nodes or foci if $b = 1$.

- ii) The singular points on the x -axis are given by $y = 0$ $cx^3 + ax^2 + \varepsilon_2x + \varepsilon_1 = 0$, so we always have a singular point far from the origin, which we do not consider in our *local* study. We may also have two singular points $q_{\pm} = (x_{\pm}, 0)$ in the neighborhood of the origin which are given approximately by $ax^2 + \varepsilon_2x + \varepsilon_1 = 0$, provided $\Delta = \varepsilon_2^2 - 4a\varepsilon_1 > 0$. The determinant of the linearized vector field is $-\sqrt{\Delta}x_+$ (resp. $\sqrt{\Delta}x_-$), at q_+ (resp. q_-). So q_+ (resp. q_-) is a saddle precisely when $x_+ > 0$, i.e. $a > 0$, $\varepsilon_2 < 0$ or $\varepsilon_1 < 0$, $\varepsilon_2 > 0$ (resp. $x_- < 0$, i.e. $a, \varepsilon_2 > 0$ or $\varepsilon_1, \varepsilon_2 < 0$).
- iii) In the case $b = 1$, there is a Hopf bifurcation at the point $q_{\pm}$ for $\varepsilon_2 = 0$, $\varepsilon_1 < 0$, leading to *attractive* (resp. *repulsive*) limit cycles if $c = 1$ (resp. $c = -1$).

In the case $b = -1$, the phase portraits are given in Figure 22. There are two topologically distinct cases, depending on the sign of a .

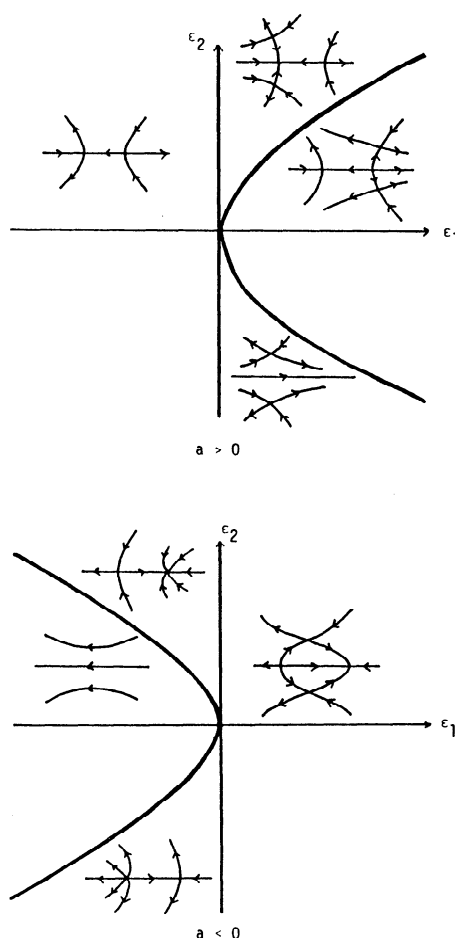


Figure 22. Bifurcation diagrams of system (6.10) for $b = -1$.

INTERPRETATION OF THE SYSTEM (6.9), WHEN $b = -1$.

Map. The singular points $q_{\pm}$ correspond to fixed points, and the points $p_{\pm}$ to periodic points of period 2.

Flow. The singular points $q_{\pm}$ correspond to periodic orbits, and the points $p_{\pm}$ to a periodic orbit of period 2.

We now study the phase portraits in the case $b = 1$, limiting ourselves to $c = 1$ (the case $c = -1$ is obtained by $x \mapsto -x$, $t \mapsto -t$, $\varepsilon_2 \mapsto -\varepsilon_2$). There are two topologically distinct cases, depending whether the points $q_{\pm}$ exist ($a > 0$) or not ($a < 0$) at the time of the Hopf bifurcation. In the two cases Hopf bifurcation occurs at $\varepsilon_2 = 0$, $\varepsilon_1 < 0$. It leads to repulsive limit cycles around $p_{\pm}$ for $\varepsilon_2 < 0$. The disappearance of the limit cycle is studied via the change of variables

$$\begin{aligned} x &= \delta u & y &= \delta v & \tau &= \delta t \\ \varepsilon_1 &= \delta^2 & \varepsilon_2 &= \delta^2 \nu_2, \end{aligned} \quad (6.13)$$

through which the system (6.9) becomes

$$\begin{aligned} u' &= 1 + au^2 + v^2 + \delta(\nu_2 + 1)u^3, \\ v' &= -2uv. \end{aligned} \quad (6.14)$$

The system at $\delta = 0$ is integrable, but not Hamiltonian, so we multiply it by the integrating factor v^{a-1} for $v \geq 0$. The region $v < 0$ is obtained by symmetry.

$$\begin{aligned} u' &= v^{a-1} + au^2 v^{a-1} + v^{a+1} + \delta(\nu_2 u v^{a-1} + u^3 v^{a-1}), \\ v' &= -2uv^a. \end{aligned} \quad (6.15)$$

For $\delta = 0$ we have a Hamiltonian system, with Hamiltonian function

$$H = \frac{v^a}{a} + u^2 v^a + \frac{v^{a+2}}{a+2}. \quad (6.16)$$

We now discuss separately the two cases $a > 0$, and $a < 0$.

- (1) $a > 0$. The Hamiltonian system has two singular points $(\pm\sqrt{1/a}, 0)$, and a heteroclinic loop through the two points for $H = 0$ (Figure 23). The limit cycle dies in the heteroclinic connection for

$$\nu_2 = -\frac{\int_{H=0} u^3 v^{a-1} dv}{\int_{H=0} uv^{a-1} dv} < 0. \quad (6.17)$$

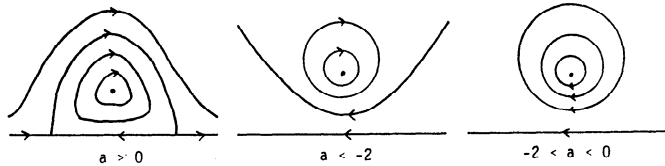


Figure 23. Closed curves of the Hamiltonian system (6.15) with $\delta = 0$, in the region $v \geq 0$.

- (2) $a < 0$. Here what really happens is more obscure. We know that the limit cycles must have disappeared before we reach the pitchfork bifurcation on the negative ε_2 -axis. The problem is that the closed curves of the Hamiltonian system become infinite, and our analysis by perturbation techniques is only valid in a compact subspace of the plane. It is evident that the limit cycles cannot disappear by themselves. There are several possibilities:

- i) They disappear by interacting with singular points at infinity.
- ii) The system (6.10) has an extra singular point on the x -axis, which is out of the range of our local study, but which could play a role in the disappearance of the limit cycles.
- ii) The limit cycles disappear by interacting with the points $(q_{\pm}, 0)$, which appear in the saddle-node bifurcation, approximately at $\varepsilon_2^2 - 4a\varepsilon_1 = 0$ $\varepsilon_2 < 0$. Zholondek [19], seems to have excluded the last possibility (see Figure 24, proposed by Zholondek), but he does not give any satisfying explanation.

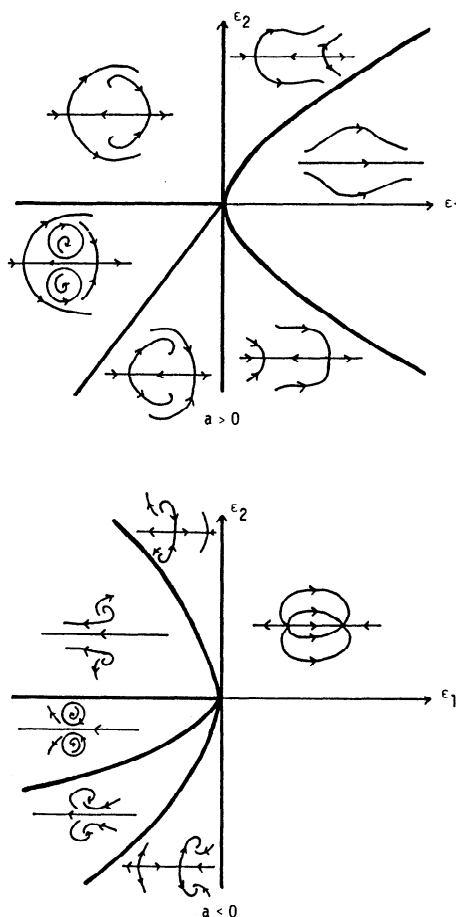


Figure 24. Bifurcation diagram of the system (6.10) for $b = 1$.

REMARK. A proof of the versality of the deformations can be found in [19]. References [5] and [10] (resp. [7]) deal with the uniqueness of the limit cycle in the case $a > 0$ (for any a).

INTERPRETATION OF THE BIFURCATION DIAGRAM OF (6.9) FOR $b = 1$.

Map. The interpretation is similar to the previous cases. The singular points $q_{\pm}$ correspond to fixed points, the singular points $p_{\pm}$ to periodic points of period 2, and the limit cycles to two closed curves of period 2, around the periodic points.

Flow. The singular points $q_{\pm}$ correspond to periodic orbits, the singular points $p_{\pm}$ to a periodic orbit of period 2, and the limit cycles to an invariant torus of period 2 (Figure 25), around the periodic orbit of period 2.

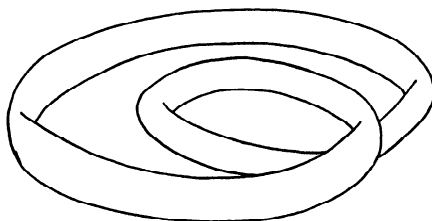


Figure 25. An invariant torus of period 2.

Appendix. Here we state the Malgrange-Weierstrass preparation theorem, and we discuss its use.

THEOREM. ([14] or [16]) Let $f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a C^∞ function in a neighborhood of the origin, such that $f(0) = 0$, $\frac{\partial^i f}{\partial x_n^i} \Big|_{x=0} = 0$ for $i < r$, and $\frac{\partial^r f}{\partial x_n^r} \Big|_{x=0} \neq 0$. Then there exist

- i) a neighborhood U of 0 in $\mathbb{R}^n$,
- ii) C^∞ functions $c_0, \dots, c_{r-1}$ depending on $x_1, \dots, x_{n-1}$, and satisfying $c_i(0) = 0$,
- iii) a C^∞ function $h: U \rightarrow \mathbb{R}$, invertible on U ,

such that we have in U

$$f(x) = h(x)P(x), \quad (\text{A.1})$$

with

$$P(x) = x_n^r + c_{r-1}(x_1, \dots, x_{n-1})x_n^{r-1} + \dots + c_1(x_1, \dots, x_{n-1})x_n + c_0(x_1, \dots, x_{n-1}).$$

REMARKS.

- (1) The functions c_i and h are in general not unique. They are unique in the case where f is analytic. In this case c_i and h are also analytic. In the C^∞ -case Mather showed that one could take the c_i 's and h as depending linearly on f .
- (2) Essentially, this theorem says that, locally, the zeros of a smooth function on the variable x_n depending on $n-1$ parameters $(x_1, \dots, x_{n-1})$, are the same as the zeros of a polynomial of degree r , under the non-degeneracy condition $\frac{\partial^r f}{\partial x_n^r} \Big|_{x=0} \neq 0$. In particular there exists a neighborhood of the origin in which the function cannot have more than r zeros in x_n for any values of $(x_1, \dots, x_{n-1})$.
- (3) The case $r = 1$ is equivalent to the implicit function theorem.

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