

EXISTENCE OF SOLUTIONS FOR AN ORIGIN-DESTINATION MATRIX ESTIMATION PROBLEM

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RÉSUMÉ. [Nguyen (1977)] considère le problème d'estimation d'une matrice origine-destination dans un contexte d'équilibre des flots sur un réseau. Nous montrons l'existence d'une solution optimale pour le problème d'optimisation qui est au coeur de son approche.

ABSTRACT. [Nguyen (1977)] considers the origin-destination matrix estimation within a network equilibrium approach. In this note we prove the existence of an optimal solution for a crucial optimization problem used by Nguyen.

1. The Problem. The problem of obtaining an O-D matrix (i.e. an origin-destination matrix) is the following: given the link flows on a network with initial nodes O_i , $i = 1, \dots, I$, and terminal nodes P_j , $j = 1, \dots, J$, (and possibly with other intermediate nodes) can one give a consistent estimate of the number of travels between node O_i and node P_j ($i = 1, \dots, I$ and $j = 1, \dots, J$). This number is the ij^{th} entry of the O-D matrix.

In [Nguyen (1977)] the author considers the following problem (P) in a network equilibrium approach to the estimation of an O-D matrix (see also [Jörnsten and Nguyen (1983)]):

$$(P) \left\{ \begin{array}{ll} \min \sum_{a \in L} \int_0^{v_a} s_a(x) dx - \sum_{i \in W} \hat{u}_i T_i & \\ \text{s.t.} & \\ T_i - \sum_{l \in Q_i} h_l = 0 & i \in W \\ h_l \geq 0 & l \in R \\ T_i \geq 0 & i \in W \\ v_a = \sum_{l \in R} \delta_{al} h_l & a \in L \end{array} \right.$$

where

L is the set of links in a finite road network,

v_a is the flow on a link a ,

W is the set of O-D pairs,

R is the set of routes,

Q_i is the set of routes between the i^{th} O-D pair

T_i is the travel demand on the i^{th} O-D pair,

h_l is the flow on route l ,

$$\delta_{al} = \begin{cases} 0 & \text{if link } a \text{ is not on route } l \\ 1 & \text{otherwise.} \end{cases}$$

Also $s_a(\cdot)$ is the travel cost function on link a : it is assumed to be continuous, strictly increasing and such that $\lim_{v_a \rightarrow \infty} s_a(v_a) = \infty$.

The question we wish to answer is whether there exists an optimal solution to problem (P) given any family of positive $\hat{u}_i$: notice that the feasible set of (P) is unbounded.

We will in fact consider the following equivalent problem:

$$(PA) \left\{ \begin{array}{l} \min \sum_{a \in L} \int_0^{\sum_l \delta_{al} h_l} s_a(x) dx - \hat{u}_i T_i \\ \text{s.t.} \\ T_i - \sum_{l \in Q_i} h_l = 0 \\ h_l \geq 0 \quad l \in R \\ T_i \geq 0 \quad (\text{redundant}) \end{array} \right.$$

The convexity of the objective function in (PA) follows readily from [Rockafellar (1970), Theorem 5.1]:

THEOREM. *Let f be a convex function from $\mathbf{R}^n$ to $(-\infty, \infty]$ and let ϕ be a convex function from $\mathbf{R}$ to $(-\infty, \infty]$ which is non-decreasing. Then $h(x) = \phi(f(x))$ is convex on $\mathbf{R}^n$ (where one sets $\phi(\infty) = \infty$).*

Indeed one can write

$$\int_0^{\sum_l \delta_{al} h_l} s_a(x) dx = \phi(f(h))$$

where

$$f: \mathbf{R}^m \rightarrow (-\infty, \infty]: h \mapsto \sum_l \delta_{al} h_l$$

(here $m = \text{card}(R)$) and

$$\phi: \mathbf{R} \rightarrow (-\infty, \infty]: t \mapsto \int_0^t s_a(x) dx.$$

Now $f(\cdot)$ is clearly convex. Since $\phi'(\cdot) = s_a(\cdot) \geq 0$, ϕ is non-decreasing (a cost function is always non-negative). To see that $\phi(\cdot)$ is convex, one has to show

$$\phi(t) - \phi(\bar{t}) \geq \nabla \phi(\bar{t})(x - \bar{x}) \quad \text{for all } x, \bar{x} \text{ in } \mathbf{R}$$

that is

$$\int_0^t s_a(x) dx \geq s_a(\bar{t})(x - \bar{x}) \quad \text{for all } x, \bar{x} \text{ in } \mathbf{R}. \quad (1)$$

Indeed if $t > \bar{t}$, then (1) becomes

$$\int_0^t s_a(x) dx \geq s_a(\bar{t})(t - \bar{t})$$

which is seen to be true from $s(x) \geq s(\bar{t})$ for $x \in [\bar{t}, t]$.

If $t < \bar{t}$, multiply (1) by -1 and proceed as above.

2. Existence of solutions. To derive the existence of an optimal solution for problem (PA) we will use the following definitions and theorem from [Rockafellar (1970)].

A convex function is *proper* if its epigraph is non-empty and contains no vertical line. A proper convex function is *closed* if it is lower semi-continuous. Let C be a non-empty convex set in $\mathbf{R}^n$, then C *recedes in the direction of* y , $y \neq 0$, iff $x + \lambda y \in C$ for every $\lambda \geq 0$ and $x \in C$. The set of all such vectors y , plus the vector zero, is the *recession cone* of C (notation O^+C). The directions in which C recedes are the *directions of recession* of C . Let f be a convex function on $\mathbf{R}^n$ not identically ∞ : the *recession function* of f (notation fO^+) is the only function such that $\text{epi}(fO^+) = O^+(\text{epi } f)$. The set of all vectors y such that $(fO^+)(y) \leq 0$ is the *recession cone* of f . The direction of vectors in this cone are the *directions of recession* of f .

THEOREM [ROCKAFELLAR (1970), THEOREM 27.3]. Let h be a closed proper convex function, and let C be a non-empty closed convex set over which h is to be minimized. If h and C have no direction of recession in common (which is trivially true if either h or C has no direction of recession at all), then h attains its infimum over C . In the case where C is polyhedral, h achieves its infimum over C under the weaker hypothesis that every common direction of recession of h and C is a direction in which h is constant.

We shall now recast our problem (PA) in the following form.

$$(PB) \begin{cases} \min(f+g)(h, T) \\ \text{s.t.} \\ (h, T) \in C \end{cases}$$

where

$$(f+g)(h, T) = \begin{cases} \sum_{a \in L} \int_0^{\sum_l \delta_{al} h_l} s_a(x) dx + \sum_i \hat{u}_i T_i & \text{if } (h, T) \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

and

$$C = \left\{ (h, T) \mid h_l \geq 0, T_i \geq 0, T_i - \sum_{l \in Q_i} h_l = 0 \text{ for all } i \right\}.$$

The set C is obviously closed.

Now $f+g$ is a closed proper convex function: we know already that it is convex and proper and since $(f+g)|_{(\mathbf{R}^n)^+}$ is continuous we have $(f+g)^{-1}([-\infty, \alpha])$ closed in $(\mathbf{R}^n)^+$ and thus in $\mathbf{R}^n$. Using [Rockafellar (1970), Theorem 7.1] we thus have that $f+g$ is closed. Let us postulate for the time being that the directions of recession of $f+g$ are of the special form $(d^h, d^T) \in \mathbf{R}^n$, where $n = \text{card}(R) + \text{card}(W)$, and where also $d^h = 0$. Now let $D = (d_1^h, d_2^T)$ be a direction of recession of C which is also a direction of recession of $f+g$: $(d_1^h, d_2^T) = (0, d_2^T)$. But this direction of recession must also be a direction of recession for each of the closed sets

$$C_i = \left\{ (h, T) \mid h \geq 0, T \geq 0, T_i - \sum_{l \in Q_i} h_l = 0 \right\} \quad i \in W.$$

But this implies that

$$(0, d_2^T) \cdot (-1, -1, \dots, -1, 0, \dots, 1, \dots, 0) = 0$$

(where the position of the 1 corresponds to the i^{th} component of T), since C_i is part of an hyperplane and directions of recession of an hyperplane must be orthogonal to its normal. Now this equality implies in turn that

$$d_{2,i}^T = 0 \quad \text{for all } i \in W.$$

Thus $D = 0$ and C and $f + g$ do not have any direction of recession in common. Thus by the theorem cited above, we obtain the existence of an optimal solution for problem (PB) and thus for problem (P).

3. Directions of recession for the function $f + g$. To complete our proof we need to show the following

PROPOSITION. *The set of direction of recession of $f + g$ is a subset of*

$$\{(d^h, d^T) \in \mathbb{R}^n \mid d^h = 0, d^T \geq 0\}.$$

PROOF: If for a certain $i \in W$ we have $d_i^T < 0$, then choosing $\lambda \in \mathbb{R}$ large enough we can make $\bar{T}_i + d_i^T < 0$ for a $(\bar{T}, \bar{h}, (f + g)(\bar{T}, \bar{h})) \in \text{epi}(f + g)$: thus it is impossible to have such a negative component in a direction of recession of $\text{epi}(f + g)$.

We are thus left with the task of showing that it is impossible to have a non-zero component of d^h . The impossibility of having $d_l^h < 0$ for a certain $l \in L$ is proved as above. So let us suppose, without loss of generality, that $d_1^h > 0$: we thus consider $(d_1^h, \dots; d_1^T, \dots) \in O^+(\text{epi}(f + g))$. But since the $s_a(\cdot)$ are positive and strictly increasing to ∞ , it is possible to choose $(\bar{h}, \bar{T})$ in the interior of $\text{dom}(f + g)$ such that

$$0 < \alpha < \sum_{a \in L} s_a \left(\sum_{l \in R} \delta_{al} \bar{h}_l \right) \delta_{a1} d_1^h.$$

Thus

$$\alpha < \left(\sum_{a \in L} s_a \left(\sum_{l \in R} \delta_{al} \bar{h}_l \right) \delta_{a1}, \dots; \hat{u}_1, \hat{u}_2, \dots \right) \cdot (d_1^h, \dots; d_1^T, \dots)$$

or

$$\alpha < \nabla(f + g)(\bar{h}, \bar{T}) \cdot (d^h, d^T).$$

But then

$$\begin{aligned} f(\bar{h}) + g(\bar{T}) + \alpha &< f(\bar{h}) + g(\bar{T}) + \nabla(f + g)(\bar{h}, \bar{T}) \cdot (d^h, d^T) \\ &\leq (f + g)((\bar{h}, \bar{T}) + (d^h, d^T)) \end{aligned}$$

since $f + g$ is convex.

Thus

$$\begin{aligned} &(\bar{h}, \bar{T}, f(\bar{h}) + g(\bar{T}) + (d_1^h, \dots; d_1^T, \dots; \alpha)) \\ &((\bar{h}, \bar{T}) + (d_1^h, \dots; d_1^T, \dots), f(\bar{h}) + g(\bar{T}) + \alpha) \notin \text{epi}(f + g) \end{aligned}$$

i.e.

$$(d^h, d^T) \notin O^+(\text{epi}(f + g)).$$

Thus $d_1^h = 0$. This ends the proof.

REFERENCES

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