

ALMOST-PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DIAGONAL OPERATORS*

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Résumé

Dans ce travail on considère l'équation différentielle abstraite $u'(t) = Au(t) + f(t)$ dans un espace de Hilbert séparable H , A étant un opérateur linéaire non borné de type diagonal et f une fonction presque-périodique de $\mathbb{R}$ dans H . On indique une condition suffisante pour assurer l'existence et l'unicité d'une solution presque-périodique ultra-faible de l'équation mentionnée.

Abstract

In this work we consider the first order differential equation $u'(t) = Au(t) + f(t)$ in a separable Hilbert space H , A being a linear, unbounded, diagonal operator, and $f(t)$ a H -valued almost-periodic function. We give a sufficient joint condition on the operator A and the function f ensuring existence and uniqueness of an almost-periodic ultra-weak solution of the above equation.

Introduction

In finite dimensional euclidean spaces, a classical result about existence and uniqueness of almost-periodic solutions refers to the system

$$\frac{du}{dt} = Au(t) + f(t), \quad u(t) = (u_1(t), \dots, u_n(t)), \quad f(t) = (f_1(t), \dots, f_n(t)), \quad A = (a_{ij})_{i,j=1}^n.$$

If the constant matrix A has no eigen-values with real part zero, and if $f(t)$ is

almost-periodic, $R \rightarrow R^n$, there exists one and only one almost-periodic solution $u(t)$ (see for inst. [1], p. 95).

For equation of the same form in Banach spaces with a bounded linear operator A having spectrum $\sigma(A)$ disjoint of the imaginary axis the same kind of result holds true (see [2], p. 88, Th. 4.4).

Finally, when the operator A is closed unbounded and generates a Co-semigroup with exponential decay a similar existence and unicity result is proved as Th. 1.1 in [4], p. 123 for strong solutions.

In the present work we restrict ourselves to a Hilbert space and the operator A , while linear unbounded, will have quite a special form, being a diagonal operator (bounded diagonal operators are discussed in Halmos [3]).

We shall present here a result about existence and uniqueness of almost-periodic ultra-weak solutions, under a condition relating together, the operator A and the given almost-periodic function f . The proofs are quite simple and the result is (probably) new.

1. Consider a separable Hilbert space H with orthonormal basis $\{e_j\}_1^\infty$. Define a linear operator A acting in H in the following way: for any e_j , Ae_j is defined and equals $\lambda_j e_j$ where λ_j are some complex numbers.

Then, for any finite linear combination $\sum_{p=1}^{\rho} \alpha_p e_p$, we put $A(\sum_{p=1}^{\rho} \alpha_p e_p) = \sum_{p=1}^{\rho} \alpha_p \lambda_p e_p$.

We have obtained this way a linear operator in H , with domain consisting of finite linear combinations of the basic vectors e_j (which is a dense set in H). Such an operator will be called diagonal.

Our first result is as follows:

LEMMA. Any element e_j belongs to $D(A^*)$ and $A^*e_j = \bar{\lambda}_j e_j$, $j = 1, 2, \dots$

— This follows from the equality: $(Ax, e_j) = (x, \bar{\lambda}_j e_j) \quad \forall x \in D(A), \quad j = 1, 2, \dots$ which is directly verifiable, taking $x = \sum_{p=1}^{\rho} \alpha_p e_p$

$$(\text{we get: } (Ax, e_j) = (\sum_{p=1}^{\rho} \alpha_p \lambda_p e_p, e_j) = \alpha_j \lambda_j; \quad (x, \bar{\lambda}_j e_j) = (\sum_{p=1}^{\rho} \alpha_p e_p, \bar{\lambda}_j e_j) = \alpha_j \lambda_j).$$

It will be used at the end, in establishing the uniqueness of the ultra-weak almost-periodic solution (in fact, of any bounded over $\mathbb{R}$ ultra-weak solution).

Next let us consider an almost-periodic function $f(t)$, $\mathbb{R} \rightarrow H$ and look for the existence of an almost-periodic function $u(t)$, which is solution of the equation $u'(t) = Au(t) + f(t)$ in the ultra-weak sense. Let be $f_j(t) = (f(t), e_j)$, a scalar-valued almost-periodic function, $j = 1, 2, \dots$ and define the almost-periodic function $u_j(t)$ which is

$$\int_{-\infty}^t e^{\lambda_j(t-s)} f_j(s) ds \quad \text{for } \operatorname{Re} \lambda_j < 0 \quad \text{or} \quad -\int_t^{\infty} e^{\lambda_j(t-s)} f_j(s) ds \quad \text{if } \operatorname{Re} \lambda_j > 0.$$

We have the obvious estimate

$$|u_j(t)| \leq \frac{1}{|\operatorname{Re} \lambda_j|} \|f_j\|_{\infty} = \gamma_j, \quad j = 1, 2, \dots, \quad \forall t \in \mathbb{R}.$$

If we assume that $\sum_{j=1}^{\infty} \gamma_j^2 < \infty$ we deduce that the numerical series $\sum_{j=1}^{\infty} |u_j(t)|^2$ is uniformly convergent over $\mathbb{R}$. Hence, the vector series in H :

$$u(t) = \sum_{j=1}^{\infty} u_j(t) e_j$$

is uniformly convergent in the norm, for $t \in \mathbb{R}$ and is accordingly an almost-periodic function, from $\mathbb{R}$ into the Hilbert space H . Now, let $v_j(t) = u_j(t) e_j$; we get $v_j'(t) = u_j'(t) e_j = (\lambda_j u_j(t) + f_j(t)) e_j = \lambda_j v_j(t) + f_j(t) e_j = A v_j(t) + f_j(t) e_j$.

Next, put $w_N(t) = v_1(t) + v_2(t) + \dots + v_N(t)$; we see that $w_N'(t) = A w_N(t) + \sum_{j=1}^N f_j(t) e_j = A w_N(t) + g_N(t)$, $t \in \mathbb{R}$, in strong sense.

Consider the class of test functions

$$K_{A^*}(\mathbb{R}) = \{\varphi(t), \quad \mathbb{R} \rightarrow D(A^*), \quad \varphi \in C_0^1(\mathbb{R}; H), \quad A^* \varphi \in C(\mathbb{R}; H)\}.$$

It follows that

$$\int_{\mathbb{R}} (w_N(t), \varphi'(t) + A^* \varphi(t))_H dt = - \int_{\mathbb{R}} (g_N(t), \varphi(t))_H dt, \quad \forall \varphi \in K_{A^*}(\mathbb{R}).$$

Previously we saw that

$$u(t) = \lim_{N \rightarrow \infty} w_N(t),$$

uniformly on $\mathbb{R}$; on the other hand, we know (see [5], pp. 98-99) that

$$f(t) = \lim_{N \rightarrow \infty} g_N(t),$$

again uniformly on $\mathbb{R}$. It results that

$$\int_{\mathbb{R}} (u(t), \varphi'(t) + A^* \varphi(t))_H dt = - \int_{\mathbb{R}} (f(t), \varphi(t))_H dt, \quad \forall \varphi \in K_{A^*}(\mathbb{R}),$$

which means that the almost-periodic function $u(t)$ is an *ultra-weak solution* of the equation: $u'(t) - Au(t) = f(t)$ on the real line.

It is also the *unique* ultra-weak almost-periodic solution on $\mathbb{R}$. This is seen as follows:

Assume that $u(t)$ is a continuous bounded over $\mathbb{R}$ function, such that

$$\int_{\mathbb{R}} (u(t), \varphi'(t) + A^* \varphi(t))_H dt = 0 \quad \forall \varphi \in K_{A^*}(\mathbb{R}).$$

Take now $\varphi(t) = v(t)e_k$, where $v(t) \in C_0^1(\mathbb{R})$. We get immediately

$$\int_{\mathbb{R}} (u(t), v'(t)e_k + v(t)A^*e_k) dt = 0, \quad k = 1, 2, \dots,$$

and consequently

$$\int_{\mathbb{R}} \{v'(t)(u(t), e_k) + v(t)(u(t), \bar{\lambda}_k e_k)\} dt = 0.$$

If $u_k(t) = (u(t), e_k)$ we get

$$\int_{\mathbb{R}} \{v'(t)u_k(t) + v(t)\lambda_k u_k(t)\} dt = 0 \quad \forall v \in C_0^1(\mathbb{R}).$$

That is

$$\frac{d}{dt} u_k(t) = \lambda_k u_k(t)$$

in sense of $D'(\mathbb{R})$. Hence, as well-known, $u_k(t) = e^{\lambda_k t} u_k(0)$, and because $u_k(t)$ is bounded over $\mathbb{R}$ while $\operatorname{Re} \lambda_k \neq 0$, it follows that $u_k(t) \equiv 0$, $\forall k = 1, 2, \dots$.

Hence $u(t) = 0$, $\forall t \in \mathbb{R}$. $\square$

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