

ON THE NUMBER OF PRIMITIVE PYTHAGOREAN TRIANGLES

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Résumé

Soit $P(N)$ le nombre de triangles Pythagoréens primitifs (c'est-à-dire de triples de nombres naturels (a,b,c) avec $a^2 + b^2 = c^2$, $a \leq b$, a,b,c relativement premiers) d'aire moins que N . Dans cet article, on établit une estimation conditionnelle (supposant que l'hypothèse de Riemann soit vraie) pour le terme résiduel $R(N)$ dans la formule asymptotique

$$P(N) = c N^{1/2} + c' N^{1/3} + R(N),$$

à savoir $R(N) = O(N^{127/560+\epsilon})$. Cela améliore les résultats de Lambek et Moser [5], de Wild [13] et de Duttlinger et Schwarz [1].

Abstract

Let $P(N)$ denote the number of primitive Pythagorean triangles (i.e., positive integer triples (a,b,c) with $a^2 + b^2 = c^2$, $a \leq b$, a,b,c coprime) of area less than N . In this paper we give a conditional estimate (subject to the Riemann hypothesis) of the error term $R(N)$ in the asymptotic

$$P(N) = c N^{1/2} + c' N^{1/3} + R(N),$$

namely $R(N) = O(N^{127/560+\epsilon})$. This improves results of Lambek and Moser [5], Wild [13] and Duttlinger and Schwarz [1].

1. Introduction

A primitive Pythagorean triangle is a triple (a, b, c) of relatively prime natural numbers with $a \leq b$ and $a^2 + b^2 = c^2$. Let $P(N)$ denote the number of primitive Pythagorean triangles with area less than N . The asymptotic behaviour of $P(N)$ was first studied by J. Lambek and L. Moser [5] who obtained

$$P(N) = c N^{1/2} + O(N^{1/3})$$

with $c = (2\pi^5)^{-1/2} \Gamma^2(1/4)$. R. Wild [13] improved this result to

$$(1) \quad P(N) = c N^{1/2} - c' N^{1/3} + R(N)$$

with $c' = |\zeta(1/3)|(1 + 2^{-1/3})/(\zeta(4/3)(1 + 4^{-1/3}))$ and $R(N) = O(N^{1/4} \log N)$. The best known bound of $R(N)$ depends on a good estimation of $M(x) := \sum_{n \leq x} \mu(n)$ (here μ denotes the Moebius function). Using $M(x) = O(x \exp(-C(\log x)^{1/2}))$, $C > 0$, J. Duttlinger and W. Schwarz [1] proved (1) with $R(N) = O(N^{1/4} \exp(-C(\log N)^{1/2}))$. The same argument together with the sharper estimate $M(x) = O(x \exp(-C(\log x)^{3/5} (\log \log x)^{-1/5}))$ (cf. [12]) leads to

$$R(N) = O(N^{1/4} \exp(-C(\log N)^{3/5} (\log \log N)^{-1/5})).$$

Assuming the correctness of the Riemann hypotheses, Duttlinger and Schwarz proved for every $\varepsilon > 0$

$$R(N) = O(N^{5/22+\varepsilon}).$$

In [6], the third named author gave the bound $R(N) = O(N^{137/604})$; this can be proved by Van der Corput's classical method for the estimation of exponential sums and by part of the analysis outlined in this paper. A stronger assertion of [6], that the exponent can be improved to 0.22679..., apparently cannot be supported by classic techniques, since the method of exponent pairs is not applicable for the complicated functions occurring in this problem. In this paper, however, we employ the new "discrete Hardy-Littlewood method" for the estimation of exponential sums (due to Bombieri, Iwaniec, Mozzochi, Huxley and Watt) to establish the following even slightly sharper result.

THEOREM. Under the assumption of the Riemann hypotheses, the asymptotic expansion (1) is true with

$$R(N) = O(N^{127/560+\varepsilon})$$

and any $\varepsilon > 0$.

2. Auxiliary results

During the entire article Landau's O -symbol and Vinogradoff's $\ll$ -notation are used (all constants involved are absolute ones). The symbol $A \cong B$ is equivalent to $A \ll B \ll A$. As usual, $e(t)$ abbreviates $e^{2\pi it}$ and $\psi(t) := t - [t] - 1/2$.

LEMMA 1 (Van der Corput, [11], p. 65). Let $g(x)$ be a real function with continuous derivatives up to the third order; $g''(x) < 0$, $m_2 \ll |g''(x)| \ll m_2$ and $g^{(3)}(x) \ll m_3$ on $[a, b]$. Then

$$\begin{aligned} \sum_{a \leq x \leq b} e(g(x)) &= e(-1/8) \sum_{\alpha \leq u \leq \beta} |g''(x(u))|^{-1/2} e(g(x(u)) - u x(u)) \\ &\quad + O(m_2^{-1/2}) + O((b-a)m_2^{1/5}m_3^{1/5}) + O(\log(2 + (b-a)m_2)), \end{aligned}$$

where $x(u)$ (for integral u) is defined by $g'(x(u)) = u$, $\alpha := g'(b)$ and $\beta := g'(a)$.

The following result is an easy consequence (derived in [9]) of Huxley's and Watt's deep estimation in [3], [4].

LEMMA 2. Let $c \in \mathbb{N}$, $M \geq 1$ and $T \geq 1$ real parameters, F a real function six times continuously differentiable on $[M/2, 2^c M]$, satisfying in this interval $|F^{(r)}| \cong M^{-r} T$ for $r = 4, 5, 6$. Suppose that $M \ll T^{4/15}$. Then for any real $M' \in [M, 2^c M]$ and any $\varepsilon > 0$,

$$\sum_{M \leq u \leq M'} e(F(u)) = O(M^{116/139} T^{9/278+\varepsilon}) + O(M^{1091/1668} T^{32/417+\varepsilon}).$$

LEMMA 3. Let the real function f be continuously differentiable on $[a, b]$ with monotone derivative $f' < 0$. Then

$$\sum_{a < x \leq b} \psi(f(x)) = \sum_{f(b) < y \leq f(a)} \psi(f^{-1}(y)) + O(\max\{|f'(a)|, |f'(b)|, \frac{1}{|f'(a)|}, \frac{1}{|f'(b)|}\}).$$

PROOF. We count the number A of lattice points inside the domain $\mathcal{D} := \{(x, y) \mid a < x \leq b, f(b) < y \leq f(x)\}$ in two different ways. Using Euler's summation formula a short calculation shows that

$$\begin{aligned} A &= \sum_{a < x \leq b} \{[f(x)] - [f(b)]\} \\ &= \text{area}(\mathcal{D}) - \sum_{a < x \leq b} \psi(f(x)) - \int_a^b f'(x) \psi(x) dx + \psi(a)(f(a) - f(b)) \\ &\quad + \psi(f(b))(b-a) + O(1). \end{aligned}$$

Interchanging f and f^{-1} (i.e., a with $f(b)$ and b with $f(a)$) this reads

$$\begin{aligned} A &= \sum_{f(b) < y \leq f(a)} \{[f^{-1}(y)] - [a]\} \\ &= \text{area}(\mathcal{D}) - \sum_{f(b) < y \leq f(a)} \psi(f^{-1}(y)) - \int_{f(b)}^{f(a)} (f^{-1})'(y) \psi(y) dy \\ &\quad + \psi(f(b))(b-a) + \psi(a)(f(a) - f(b)) + O(1). \end{aligned}$$

Now f' has been assumed to be monotone, hence the second mean value theorem can be applied to estimate the first integral by $\max\{|f'(a)|, |f'(b)|\}$ and the second one by $\max\{1/|f'(a)|, 1/|f'(b)|\}$. Comparing both results completes the proof of the lemma.

3. Proof of our theorem

J. Lambek and L. Moser [5] reduced the problem of calculating $P(N)$ to that of counting the number $L(N)$ of lattice points inside the planar domain

$$\mathcal{D} := \{(x, y) \mid x y(x^2 - y^2) < N, 0 < y < x\} \quad (N \in \mathbb{R}^+).$$

They established the relation

$$(2) \quad P(N) = \sum_{k \geq 0} (-1)^k L'(N 4^{-k}),$$

where $L'(N)$ is the number of primitive lattice points in $\mathcal{D}$. To estimate $L(N)$,

we use an asymptotic result of Duttlinger and Schwarz ([1], pp. 369-370):

$$(3) \quad L(N) = d N^{1/2} - d' N^{1/3} - F_1 + O(N^{2/15}),$$

with

$$(4) \quad F_1 = \sum_{\substack{y < Y \\ y \in \mathbb{N}}} \psi(\xi(y)) + \sum_{\substack{y < Y \\ y \in \mathbb{N}'}} \psi\left(\frac{1}{\sqrt{2}} \xi(y)\right) + \sum_{\substack{y < Y \\ y \in \mathbb{N}''}} \psi\left(\frac{1}{\sqrt{2}} \xi(y) + \frac{1}{2}\right).$$

Here the abbreviations

$$\begin{aligned} d &= (4\sqrt{2}\pi)^{-1} \Gamma^2(1/4) & \mathbb{N}' &= \sqrt{2} \mathbb{N} \\ d' &= |\zeta(1/3)| (1 + 2^{-1/3}) & \mathbb{N}'' &= \sqrt{2} \mathbb{N} - \sqrt{2}/2 \\ Y &= \sqrt{2} \sin(\pi/8) N^{1/4} \end{aligned}$$

have been used. For $0 < y \leq Y$ the function $\xi(y)$ is uniquely defined by

$$(5) \quad \xi^3 y - \xi y^3 - N = 0.$$

By Cardano's formula,

$$\xi(y) = \left(\frac{N}{2y} + \sqrt{D}\right)^{1/3} + \left(\frac{N}{2y} - \sqrt{D}\right)^{1/3}, \quad D = \frac{1}{4} \left(\frac{N}{y}\right)^2 - \frac{1}{27} y^6,$$

in particular, $\xi(y) > y$ and $\xi(Y) = (\sqrt{2} + 1)Y$. It remains to estimate sums of the form

$$S_{a,b} := \sum_{\substack{y < Y \\ y \in (aN-ab)}} \psi\left(\frac{1}{a} \xi(y) + b\right) = \sum_{\substack{t < Y' \\ t \in \mathbb{N}}} \psi(f(t))$$

where $Y' := Y/a + b$, $f(t) := (at-ab)/a + b$ and (a,b) is one of the pairs $(1,0)$, $(\sqrt{2},0)$ or $(\sqrt{2}, 1/2)$. Introducing a constant $\alpha \in (0,1)$ (which will be fixed later on) we split $S_{a,b}$ into two parts:

$$(6) \quad S_{a,b} = \sum_{t < \alpha Y'} \psi(f(t)) + \sum_{\alpha Y' \leq t < Y'} \psi(f(t)) =: S_1 + S_2.$$

Applying the inequalities of Erdős/Turán and Koksma (cf. [2], p. 104 and p. 107) to the sequence $f(t)$, $t \in \mathbb{N}$, $t < \alpha Y'$, and the function ψ , we obtain

$$(7) \quad S_1 \ll Y H^{-1} + \sum_{h=1}^H \frac{1}{h} |S(h)|,$$

where $H = [N^{13/152}]$ ($h \leq H$ throughout the sequel) and

$$S(h) := \sum_{t < \alpha Y'} e(-h f(t)).$$

To estimate $S(h)$, the summation interval $[1, \alpha Y')$ is divided into J subintervals of the form $[t_j, t_{j+1})$, with $t_0 := 1$, $t_j := U 2^j$ for $j := 1, \dots, J-1$ and $t_J := \alpha Y'$ ($2t_{J-1} < \alpha Y' \leq 4t_{J-1}$, $U > 1$ is a free real parameter). The first partial sum ($j = 0$) is trivially estimated by U and for $j \geq 1$ we use Lemma 1 to transform each of the sums. Implicit differentiation of (5) leads to

$$\begin{aligned} \xi'(y) &= -z(z^2-3)(3z^2-1)^{-1} \\ (8) \quad \xi''(y) &= 12zy^{-1}(z^2+1)^2(z^2-1)(3z^2-1)^{-3} \\ \xi'''(y) &= -12zy^{-2}(z^4-1)(21z^6-49z^4+63z^2+5)(3z^2-1)^{-5}, \end{aligned}$$

where $z = z(y) = \xi(y)/y > 1$ and $z'(y) = -4zy^{-1}(z^2-1)(3z^2-1)^{-1} < 0$. Therefore $z(y)$ is a monotone decreasing function with $z(y) \geq z(Y) = \sqrt{2} + 1$. From the explicit representation of ξ we get

$$(9) \quad \xi(y) \cong N^{1/3} y^{-1/3} \quad \text{and} \quad z(y) \cong N^{1/3} y^{-4/3}.$$

Hence the assumptions of Lemma 1 are verified with

$$\begin{aligned} g(t) &:= -h f(t) = -\frac{h}{a} \xi(at-ab) - hb, \\ m_2 &:= h N^{1/3} t_j^{-7/3}, \quad m_3 := h N^{1/3} t_j^{-10/3}, \end{aligned}$$

and we obtain

$$\begin{aligned} \sum_{t_j < t \leq t_{j+1}} e(-h f(t)) &= e(-1/8) \sum_{u_{j+1} < u \leq u_j} |g''(t(u))|^{-1/2} e(F(u)) \\ (10) \quad &+ O((h N^{1/3})^{-1/2} t_j^{7/6}) + O((h N^{1/3})^{2/5} t_j^{-2/5}) \\ &+ O(\log N) \end{aligned}$$

with $u_j := -h f'(t_j)$, $t(u) := y(u)/a + b$, $F(u) := g(t(u)) - ut(u)$. The function $y(u)$ is defined by $\xi'(y(u)) = -u/h$.

We want to estimate this new exponential sum by the method of exponent pairs; for this purpose we need a somewhat closer analysis of the functions involved. By the explicit formula for $\xi(y)$ and the above definition, it follows that

$$(11) \quad g(t) = -hb - ha^{-4/3}(t-b)^{-1/3}N^{1/3}G(a(t-b)N^{-1/4}),$$

where the function G is analytic near 0 and $G(0) \neq 0$. By Leibniz's rule, for $k \geq 1$,

$$(12) \quad \begin{aligned} g^{(k)}(t) &= -a^{-4/3}G(0)C_k h(t-b)^{-1/3-k} N^{1/3}(1 + O((t-b)N^{-1/4})) \\ &= -a^{-4/3}G(0)C_k h N^{1/3} t^{-1/3-k}(1 + o(1)), \end{aligned}$$

for t^{-1} and $t N^{-1/4}$ small; here $C_k = \prod_{m=0}^{k-1} (-\frac{1}{3} - m)$. From this we readily verify that our function $g(t)$ satisfies condition (3) in [10], p. 214 (with $s = 4/3$), provided that the constant α in (6) is sufficiently small and U is sufficiently large (since, in this context, $U \leq t \leq \alpha Y'$). By Lemma 4 in [10], the same is true for $F(u)$, with $\sigma = 3/4$.

Recently Huxley and Watt [4] have proved that, for any $\varepsilon' > 0$, $(9/56 + \varepsilon', 37/56 + \varepsilon')$ is an exponent pair. Applying the "A-step" twice, we get the exponent pair $(9/278 + \varepsilon, 241/278 + \varepsilon)$, $\varepsilon > 0$. From this we infer for any $M' \in (u_{j+1}, u_j]$ (recalling that $u_j \approx u_{j+1} \approx h z \approx h N^{1/3} t_j^{-4/3}$) that

$$(13) \quad \sum_{u_{j+1} < u \leq M'} e(F(u)) \ll (h N^{1/3})^{241/278+\varepsilon} t_j^{-937/834}.$$

We apply summation by parts in (10) (noting that $|g''(t(u))|^{-1/2}$ has only $O(1)$ maxima and minima and is of order $O(h^{-1/2} N^{-1/6} t_j^{7/6})$ for $t_j \leq t(u) \leq t_{j+1}$, $j \geq 1$) to infer that

$$\begin{aligned} \sum_{t_j < t \leq t_{j+1}} e(-h(f(t))) &\ll (h N^{1/3})^{51/139+\varepsilon} t_j^{6/139} + (h N^{1/3})^{-1/2} t_j^{7/6} \\ &\quad + (h N^{1/3})^{2/5} t_j^{-2/15}. \end{aligned}$$

Recalling, that (t_j) is a geometric sequence with $U < t_j \ll N^{1/4}$, we arrive at

$$S(h) = \sum_{j=0}^{J-1} \sum_{t_j < t \leq t_{j+1}} e(-h f(t)) \ll U + h^{51/139+\varepsilon} N^{37/278+\varepsilon} + (h N^{1/3})^{2/5} U^{-2/15}.$$

The choice $U := h^{6/17} N^{2/17}$ and $H := [N^{13/152}]$ together with (7) finally yields

$$S(h) << h^{51/139+\epsilon} N^{37/278+\epsilon}$$

and (with a new $\epsilon > 0$)

$$(14) \quad S_1 << N^{25/152+\epsilon}.$$

To estimate S_2 we first use Lemma 3 to transform the ψ -sum (note, that $1 << f'(Y') < f'(\alpha Y') << 1$):

$$(15) \quad S_2 = \sum_{A < x \leq B} \psi(f^{-1}(x)) + O(1).$$

Here $A := f(Y') \cong Y$ and $B := f(\alpha Y') \cong Y$ are of the same order $Y \cong N^{1/4}$. Like in (7), the inequalities of Erdős/Turán and Koksma are applied to get

$$(16) \quad S_2 << N^{1/4} H^{-1} + \sum_{h=1}^H \frac{1}{h} |S'(h)|$$

with

$$S'(h) := \sum_{A < t \leq B} e(-h f^{-1}(t)),$$

and the exponential sum $S'(h)$ is transformed by Lemma 1. To calculate the derivatives of $f^{-1}(t) = \xi^{-1}(x(t))/a + b$ where $x(t) := at - ab$, we use the defining equation of $\xi^{-1}(x)$, which is, up to the sign of N , the same as that for $\xi(y)$:

$$(\xi^{-1})^3 x - (\xi^{-1}) x^3 + N = 0.$$

Hence the derivatives of ξ^{-1} are of the form (8), if y is replaced by x and z by $w = w(x) = \xi^{-1}(x)/x$. The function $w(x)$ is monotone decreasing with range $(w(\infty), w(\xi(Y))) = (0, \sqrt{2} - 1]$; furthermore (9) yields

$$\xi^{-1}(x) \cong N x^{-3}, \quad w(x) \cong N x^{-4}.$$

With the definitions

$$g(t) := -h f^{-1}(t)$$

$$m_2 := h N^{-1/4}, \quad m_3 := h N^{-1/2}$$

$$A' := -h(f^{-1})'(A) \cong h, \quad B' := -h(f^{-1})'(B) \cong h,$$

Lemma 1 leads to

$$(17) \quad S'(h) = e(-1/8) \sum_{B' < u \leq A'} |g''(t(u))|^{-1/2} e(\hat{F}(u)) + O(h^{-1/2} N^{1/8}) \\ + O(h^{2/5} N^{1/10}).$$

The function $t(u) := x(u)/a + b$ is defined by $(\xi^{-1})'(x(u)) = -u/h$ and $\hat{F}(u) := g(t(u)) - u t(u)$. Starting from

$$\hat{F}'(u) = -t(u) = -x(u)/a - b \quad \text{and} \quad (\xi^{-1})''(x(u))x'(u) = -1/h,$$

elementary calculations (which had been done by a computer program for symbolic computation) yield (for $r = 4, 5, 6$)

$$\hat{F}^{(r)}(u) = (-1)^r x(u) a^{-1} (hw)^{1-r} (3w^2-1)^{r+1} (w^2+1)^{-3r+4} (w^2-1)^{-r+1} P_{3r-6}(w^2) b_r$$

with

$$P_6(x) = 693x^6 - 5082x^5 + 14259x^4 - 14476x^3 + 6923x^2 + 518x + 45,$$

$$P_9(x) = 6237x^9 - 93555x^8 + 481572x^7 - 1114092x^6 + 1394582x^5 - 868378x^4 \\ + 243124x^3 + 8708x^2 + 1589x + 117,$$

$$P_{12}(x) = 355509x^{12} - 9006228x^{11} + 75001626x^{10} - 293761188x^9 + 651328587x^8 \\ - 860670888x^7 + 689351852x^6 - 308477288x^5 + 64183707x^4 \\ - 437444x^3 + 245210x^2 + 31500x + 1989,$$

$$b_4 = 1/1728, \quad b_5 = 5/20736, \quad b_6 = 5/248832,$$

and $w = w(x(u))$. It is readily verified that $P_r(w^2) > 0$ for $r = 4, 5, 6$ and $0 \leq w^2 \leq 3 - 2\sqrt{2}$, hence

$$|\hat{F}^{(r)}(u)| \cong N^{1/4} h^{1-r}.$$

Lemma 2 is applicable with $M := 2B' \cong h$, $T := N^{1/4}h$ and $M' \leq A'$:

$$\sum_{B' \leq u \leq M'} e(\hat{F}(u)) \ll h^{241/278+\epsilon} N^{9/1112+\epsilon} + h^{1219/1668+\epsilon} N^{8/417+\epsilon}.$$

Applying summation by parts, we obtain

$$S'(h) \ll h^{51/139+\epsilon} N^{37/278+\epsilon} + h^{385/1668+\epsilon} N^{481/3336+\epsilon} + h^{2/5} N^{1/10}.$$

If we enter this into (16) and use $H := [N^{13/152}]$, the estimate

$$S_2 \ll N^{25/152+\epsilon}$$

follows. Combining this with (3), (4), (6) and (14) yields

$$(18) \quad L(N) = d N^{1/2} - d' N^{1/3} + O(N^{25/152} \epsilon).$$

By the argument from [8], Theorem 1 (with a second main term), we thus infer that, if the Riemann hypotheses is true,

$$L'(N) = (d/\zeta(2))N^{1/2} - (d'/\zeta(4/3))N^{1/3} + O(N^{127/560+\epsilon}).$$

Now the theorem immediately follows from (2).

CONCLUDING REMARK. Our result can in fact be stated in a slightly more precise form, namely

$$R(N) = O(N^{127/560} \exp(c(\log N)^{0.6393})) \quad (c > 0).$$

This follows from the recently proved conditional bound for $\sum_{n \leq x} \mu(n)$ due to H. H. Maier and H. Montgomery [7], together with the observation that in Huxley's and Watt's estimates all ϵ 's can be replaced by logarithmic factors.

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