

ON RENEWAL PROCESSES HAVING STABLE INTER-RENEWAL INTERVALS AND STABLE REWARDS*

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Résumé

Mandelbrot, et plus tard Taqqu et Lévy, ont étudié des processus stationnaires de récompense et de renouvellement qui donnent lieu à des processus auto-similaires. Nous généralisons le modèle en considérant d'une part des récompenses qui appartiennent au domaine d'attraction d'une variable aléatoire stable d'indice $0 < \beta \leq 2$, et d'autre part, des temps de renouvellement qui appartiennent au domaine d'attraction d'une variable aléatoire stable d'indice $1 < \alpha \leq 2$.

1. Introduction

This paper generalizes previous work of Mandelbrot [M] and Taqqu and Levy [T-L] by examining stationary renewal processes with rewards that belong to $\mathcal{D}(\beta)$ with $0 < \beta \leq 2$, that is, which belong to the domain of attraction of a non-Gaussian (strictly) stable distribution with index β . The inter-renewal times are assumed to belong to $\mathcal{D}(\alpha)$ with $1 < \alpha \leq 2$. The Gaussian case $\alpha = \beta = 2$, which is omitted here, has been studied in [T-L].

To each inter-renewal interval we associate a reward. This enables us to define processes $W = W(t)$ and $V = V(t)$, indexed by $t = 1, 2, \dots$ as follows. On

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each inter-renewal interval, $W(t)$ is constant and equal to the reward on that interval. On the other hand, $V(t)$ is 0 in the inter-renewal interval, but at the renewal epoch at the end of the interval, $V(t)$ assumes a value equal to the reward times the length of the interval. Thus, sums of $W(t)$ and $V(t)$ on inter-renewal intervals are equal. (This correspondence between $W(t)$ and $V(t)$ fails, however, for the first inter-renewal interval. See Section 2 for more details.)

Consider now independent and identically distributed (i.i.d.) copies $W_m(t)$ and $V_m(t)$, $m = 1, 2, \dots$ of $W(t)$ and $V(t)$, respectively, and let $X_m(t)$ denote either $W_m(t)$ or $V_m(t)$. Our objective is to determine the limit

$$L - \lim_{\substack{M \rightarrow \infty \\ T \rightarrow \infty}} \frac{1}{A(T, M)} \sum_{t=1}^{[Ty]} \sum_{m=1}^M X_m(t), \quad 0 \leq y \leq 1,$$

where $L - \lim$ stands for convergence of the finite-dimensional distributions and $A(T, M)$ are suitable normalization constants. The precise nature of the limit depends on whether

- (i) $\beta < \alpha$ or $\alpha < \beta$
- (ii) $X = W$ or $X = V$
- (iii) $T \rightarrow \infty$ before $M \rightarrow \infty$, or $M \rightarrow \infty$ before $T \rightarrow \infty$.

For example (Theorem 1), under $\lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty}$, the limiting process is the Lévy-stable motion with index $\rho = \min(\alpha, \beta)$, whether $X = W$ or $X = V$. The Lévy-stable motion is a stable process with stationary independent increments.

In all cases the limit must be a *self-similar process*. A process $\Lambda(y)$ is self-similar with parameter H if the finite-dimensional distributions of $\Lambda(y)$ and $a^{-H} \Lambda(y)$ are the same for all $a > 0$. For example, the Lévy-stable motion with index ρ is self-similar with parameter $1/\rho$.

The case $\alpha < \beta$, $X = W$ and $\lim_{T \rightarrow \infty} \lim_{M \rightarrow \infty}$ is not treated here. In the case $\beta = 2$ the limit turned out to be *fractional Brownian motion*, a Gaussian process with stationary increments, self-similar with $H = \frac{\beta - \alpha + 1}{\beta} = \frac{3 - \alpha}{2}$ [T-L]. When $\beta < 2$, the limit is a stable process of index β with stationary increments and also self-

similar with $H = \frac{\beta - \alpha + 1}{\beta}$. However, the limit may not be the *fractional Lévy motion*, as defined in [T] or [T-W]. The results concerning this case will appear elsewhere.

Section 2 contains the precise assumptions and definitions of the processes, while the statements of the results are in Section 3. Proofs are in Section 4.

2. Assumptions and definitions

The basic ingredient is a stationary renewal-reward process with inter-renewal intervals of length U_k , $k = 1, 2, \dots$ and corresponding rewards W_k , $k = 1, 2, \dots$. Random variables U_0 and W_0 are also needed. The following assumptions on the sequences $\{U_k, k \geq 0\}$ and $\{W_k, k \geq 0\}$ will hold throughout the paper:

(A.1) The sequence $\{W_k, k \geq 0\}$ is i.i.d., and either $EW_0^2 < \infty$ (case $\beta = 2$) or if $0 < \beta < 2$, then

$$P(W_0 \leq -x) \sim c^- x^{-\beta} g(x), \quad P(W_0 > x) \sim c^+ x^{-\beta} g(x) \quad \text{as } x \rightarrow \infty$$

for some constants $c^- \geq 0$, $c^+ \geq 0$, such that $c^- + c^+ > 0$ and for some slowly varying function $g(x)$. Moreover,

if $1 < \beta < 2$, then $EW_0 = 0$;

if $\beta = 1$, then W_0 is symmetric.

(A.2) The sequence $\{U_k, k \geq 0\}$ is i.i.d. and either $EU_0^2 < \infty$ (case $\alpha = 2$) or, if $1 < \alpha < 2$, then $P\{U_0 > x\} \sim x^{-\alpha} h(x)$ as $x \rightarrow \infty$ where $h(x)$ is a slowly varying function. Let $\mu = EU_0$.

(A.3) $\{W_k, k \geq 0\}$ and $\{U_k, k \geq 0\}$ are independent.

REMARK. Assumption (A.1) implies that W_0 is in the domain of attraction of a stable random variable of index $0 < \beta \leq 2$, that is, there exists a sequence of constants $A_n(\beta)$ such that

$$L = \lim_{n \rightarrow \infty} A_n(\beta)^{-1} \sum_{k=1}^n W_k \stackrel{\Delta}{=} \Lambda_\beta.$$

The limiting random variable is normal when $\beta = 2$. When $0 < \beta < 2$, $A_n(\beta) = n^{1/\beta} L_g(n)$ where L_g is a slowly varying function satisfying for $x \neq 0$

$$\lim_{n \rightarrow \infty} L_g(n)^{-1/\beta} g(n^{1/\beta} L_g(n) |x|) = 1.$$

The stable random variable Λ_β satisfies

$$P(\Lambda_\beta \leq -x) \sim c^- x^{-\beta}, \quad P(\Lambda_\beta > x) \sim c^+ x^{-\beta}$$

as $x \rightarrow \infty$, and its distribution is totally characterized by β and the constants of asymptoticity c^- and c^+ (see for example [I-L]).

A similar remark holds for U_k with β, W_k, g, L_g replaced by $\alpha, U_k - \mu, h, L_h$ respectively and with $c^- = 0, c^+ = 1$.

NOTATION. We shall write

$$(2.1) \quad \rho = \min(\alpha, \beta), \quad L_\rho = \begin{cases} L_g & \text{if } \beta < \alpha \\ L_h & \text{if } \alpha > \beta. \end{cases}$$

To get a *stationary* renewal process, define the time S_0 of the first renewal to be independent of $\{U_k, k \geq 0\}$, with

$$(2.2) \quad P(S_0 = x) = \frac{1}{\mu} P(U_0 > x), \quad x = 0, 1, 2, \dots,$$

and form the partial sums

$$S_k = S_0 + \sum_{j=1}^k U_j, \quad k = 1, 2, \dots$$

Then $\{S_k, k \geq 0\}$ is a stationary renewal process with first inter-renewal interval S_0 and subsequent inter-renewal intervals $U_k, k \geq 1$. Note that S_0 does not have the same distribution as U_0 .

The processes $W = \{W(t), t \geq 1\}$ and $V = \{V(t), t \geq 1\}$ can now be defined as follows:

$$W(t) = \sum_{k=0}^t W_k I(S_{k-1} < t \leq S_k) = W_k \quad \text{if } S_{k-1} < t \leq S_k, \quad (2.3)$$

$$V(t) = \sum_{k=0}^t U_k W_k I(S_k = t) = \begin{cases} U_k W_k & \text{if } t = S_k \\ 0 & \text{if } t \neq S_k. \end{cases}$$

Here $S_{-1} = 0$. Thus, $\sum_{t \in (S_{k-1}, S_k]} W(t)$ and $\sum_{t \in (S_{k-1}, S_k]} V(t)$ are both equal to $U_k W_k$ when $k \geq 1$. These sums are different when $k = 0$ because the first equals $S_0 W_0$ while the second equals $U_0 W_0$. This difference has important consequences.

Observe also that the tail probabilities of $U_0 W_0$, are controlled by W_0 when $\beta < \alpha$, and by U_0 when $\alpha < \beta$. For instance, as $x \rightarrow \infty$,

$$(2.4) \quad P(|U_0 W_0| \geq x) = \begin{cases} (c^- + c^+) E U_0^\beta x^{-\beta} g(x) & \text{if } \beta < \alpha \\ E |W_0|^\alpha x^{-\alpha} h(x) & \text{if } \alpha < \beta. \end{cases}$$

Thus $U_k W_k \in \mathcal{D}(\rho)$ where $\rho = \min(\alpha, \beta)$. See Proposition 4.1 below for more details.

REMARK. For simplification, we consider here a one-sided stationary renewal sequence $\{S_k, k \geq 0\}$. However, the results of this paper continue to hold if $\{S_k, -\infty < k < \infty\}$ is a two-sided stationary sequence and $W(t)$ and $V(t)$ are defined as in [T-L] for $t = \dots, -1, 0, 1, \dots$.

3. Statement of the results

Let $W_m(t)$ and $V_m(t)$, $m = 1, 2, \dots$ be i.i.d. copies of $W(t)$ and $V(t)$ respectively and let $X_m(t)$ denote either $W_m(t)$ or $V_m(t)$. We are interested in the behavior of the cumulative sum process.

$$X^*(Ty, M) = \sum_{t=1}^{[Ty]} \sum_{m=1}^M X_m(t), \quad 0 \leq y \leq 1,$$

suitably normalized as $T \rightarrow \infty$ and $M \rightarrow \infty$.

The first result holds for $\alpha \neq \beta$ and for either $X = W$ or $X = V$. It involves $\lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty}$. We use the notation $W_0^+ = \max(W_0, 0)$ and $W_0^- = -\min(W_0, 0)$, and also L_ρ defined in (2.1). The notation " $\stackrel{\Delta}{=}$ " denotes equality of the finite

THEOREM 1. If $\alpha \neq \beta$ and $\rho = \min(\alpha, \beta)$, then if either $X = W$ or $X = V$,

$$(3.1) \quad L - \lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{M^{1/\rho}} \frac{1}{T^{1/\rho} L_\rho(T)} X^*(Ty, M) \stackrel{\Delta}{=} \mu^{-1/\rho} \Lambda_\rho(y), \quad 0 \leq y \leq 1,$$

where $\Lambda_\rho(y)$ is a driftless Lévy-stable motion: it has stationary independent increments, satisfies $\Lambda_\rho(0) = 0$, and the random variable $\Lambda_\rho \equiv \Lambda_\rho(1)$ satisfies as $x \rightarrow \infty$,

$$(3.2a) \quad P(\Lambda_\rho \leq -x) \sim c^- (E U_0^\beta) x^{-\beta}, \quad P(\Lambda_\rho \geq x) \sim c^+ (E U_0^\beta) x^{-\beta} \quad \text{if } \rho = \beta,$$

$$(3.2b) \quad P(\Lambda_\rho \leq -x) \sim E(W_0^-)^\alpha x^{-\alpha}, \quad P(\Lambda_\rho \geq x) \sim E(W_0^+)^\alpha x^{-\alpha} \quad \text{if } \rho = \alpha.$$

To see why $\Lambda_\rho(y)$ appears as a limit under $\lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty}$, we consider first the single process (e.g. $M = 1$) $X^*(T) = X^*(T, 1)$, and we define the number of renewals in the closed interval $[0, T]$ by

$$(3.3) \quad \begin{aligned} K(T) &= \# \text{ of integers } k \geq 0 \text{ satisfying } S_k \leq T \\ &= \begin{cases} 1 + \sup\{0 \leq k \leq [T]: S_k \leq T\} & \text{if } S_0 \leq T \\ 0 & \text{if } S_0 > T. \end{cases} \end{aligned}$$



Now for large values of T , $X^*(T)$ for $X = W$ or $X = V$ behaves asymptotically

like $\sum_{k=1}^{K(T)} U_k W_k \stackrel{\Delta}{=} \mu^{-1/\rho} U_k W_k$, since $\lim_{T \rightarrow \infty} \frac{K(T)}{T} = \mu^{-1}$ (with probability one). But $U_k W_k \in \mathcal{D}(\rho)$ (see (2.4)). Consequently, $L - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} X^*(T) \stackrel{\Delta}{=} \mu^{-1/\rho} \Lambda_\rho$. On the other hand, taking the sum $\sum_{m=1}^M \Lambda_\rho^{(m)}$ of M i.i.d. copies $\{\Lambda_\rho^{(m)}, 1 \leq m \leq M\}$ of Λ_ρ yields a stable distribution $M^{1/\rho} \Lambda_\rho$ so that the conclusion under the second limit, $\lim_{M \rightarrow \infty}$, is obvious.

The next result states that the same limit is obtained as in Theorem 1 under $\lim_{T \rightarrow \infty} \lim_{M \rightarrow \infty}$, but only if $\beta < \alpha$.

THEOREM 2. If $\beta < \alpha$, then for either $X = W$ or $X = V$

$$(3.4) \quad L - \lim_{T \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{T^{1/\beta}} \frac{1}{M^{1/\beta} L_g(M)} X^*(Ty, M) \stackrel{\Delta}{=} \mu^{-1/\beta} \Lambda_\beta(y)$$

where $\Lambda_\beta(y)$ is the Lévy-stable motion of Theorem 1 with $\rho = \beta$.

REMARK. We prove here only the case $X = V$. The proof for $X = W$ will appear elsewhere.

To understand (3.4), observe that the process arising from the first limit, $Z_{X,T}(y) \stackrel{\Delta}{=} L - \lim_{M \rightarrow \infty} M^{-1/\beta} L_g(M)^{-1} X^*(Ty, M)$, must have probability tails that imitate those of $X^*(Ty)$, in that for each T , as $x \rightarrow \infty$,

$$P(|Z_{X,T}(y)| \geq x) \sim d_{X,T}(y) x^{-\beta} \quad \text{if} \quad P(|X^*(Ty)| \geq x) \sim d_{X,T}(y) x^{-\beta} g(x).$$

When $X = V$, the process $Z_{V,T}(y)$, $0 \leq y \leq 1$, has independent increments while the process $Z_{W,T}(y)$, $0 \leq y \leq 1$, has stationary non-independent increments. In either case, we shall see that $d_{X,T}(y) \sim d_0 Ty$ as $T \rightarrow \infty$ for some $d_0 > 0$, irrespective of whether $X = W$ or $X = V$.

The case $\alpha < \beta$ yields a result different from that of Theorem 2 and, in fact, we must distinguish between the case $X = W$ and $X = V$. The result for $X = V$ is

THEOREM 3. If $\alpha < \beta$ and $X = V$ then

$$(3.5) \quad L - \lim_{T \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{M^{1/\alpha} L_h(M)} V^*(Ty, M) \stackrel{\Delta}{=} \Lambda_{\alpha,+}(y), \quad 0 \leq y \leq 1,$$

where $\Lambda_{\alpha,+}(0) = 0$, and for $y \neq 0$, $\Lambda_{\alpha,+}(y)$ is a constant process equal to a stable random variable $\Lambda_{\alpha,+}$ that satisfies

$$(3.6) \quad P(\Lambda_{\alpha,+} \leq -x) \sim (1-\mu^{-1})E(W_0^-)^{\alpha} x^{-\alpha}, \quad P(\Lambda_{\alpha,+} \geq x) \sim (1-\mu^{-1})E(W_0^+)^{\alpha} x^{-\alpha}$$

as $x \rightarrow \infty$.

REMARK. When $\alpha < \beta$ and $X = W$, the corresponding version of (3.5) yields a stable process of index β with stationary increments but having self-similarity

parameter $H = \frac{\beta - \alpha + 1}{\beta}$. The character of the finite-dimensional distributions are still being explored and will appear in future work.

4. Proofs of Theorems 1, 2 ($X = V$) and 3

We will use the following symbolic notation. For any non-zero $a \in \mathbb{R}$, let

$$\sigma(a) = \begin{cases} + & \text{if } a > 0 \\ - & \text{if } a < 0 \end{cases}, \quad -\sigma(a) = \begin{cases} - & \text{if } a > 0 \\ + & \text{if } a < 0. \end{cases}$$

Thus, for example, if $a > 0$,

$$\begin{aligned} W^{\sigma(a)} &= W^+ \equiv \max(0, W), & c^{\sigma(a)} &= c^+, \\ W^{-\sigma(a)} &= W^- \equiv -\min(0, W), & c^{-\sigma(a)} &= c^-. \end{aligned}$$

In section 3, we indicated that the tail probabilities of $U_0 W_0$ are controlled by W_0 when $\beta < \alpha$ and by U_0 when $\alpha < \beta$. The following proposition gives the precise asymptotic behavior of the tails.

PROPOSITION 4.1. Let $U_0 \geq 0$, W_0 , c^+ and c^- be defined as in Section 2.

Then as $x \rightarrow \infty$

(i) if $\beta < \alpha$

$$\begin{aligned} P(aU_0 W_0 \leq -x) &\sim |a|^\alpha c^{-\sigma(a)} E U_0 x^{\beta-\beta} g(x), \\ P(aU_0 W_0 \geq x) &\sim |a|^\alpha c^{\sigma(a)} E U_0 x^{\beta-\beta} g(x). \end{aligned}$$

(ii) if $\alpha < \beta$

$$\begin{aligned} P(aU_0 W_0 \leq -x) &\sim |a|^\alpha E(W_0^{-\sigma(a)})^\alpha x^{-\alpha} h(x), \\ P(aU_0 W_0 \geq x) &\sim |a|^\alpha E(W_0^{\sigma(a)})^\alpha x^{-\alpha} h(x). \end{aligned}$$

PROOF. We only consider the case (ii) and $P\{aU_0 W_0 \leq -x\}$. Let F_{W_0} be the distribution of W_0 . If $a > 0$, then by (A.3) and $U_0 \geq 0$,

$$\begin{aligned}
P\{aU_0 W_0 \leq -x\} &= \int_{-\infty}^{+\infty} P(U_0 \geq \frac{x}{a(-w)}) dF_{W_0}(w) \\
&= \int_{-\infty}^0 P(U_0 \geq \frac{x}{a(-w)}) dF_{W_0}(w) \\
&\sim x^{-\alpha} h(x) a^\alpha \int_{-\infty}^0 (-w)^\alpha dF_{W_0}(w) \\
&= x^{-\alpha} h(x) |a|^\alpha E(W^{-\sigma(a)})^\alpha.
\end{aligned}$$

If $a < 0$,

$$\begin{aligned}
P(aU_0 W_0 \leq -x) &= \int_0^\infty P(U_0 \geq \frac{x}{(-a)w}) dF_{W_0}(w) \\
&\sim x^{-\alpha} h(x) (-a)^\alpha E(W_0^+)^\alpha \\
&= x^{-\alpha} h(x) |a|^\alpha E(W_0^{-\sigma(a)})^\alpha. \quad \square
\end{aligned}$$

Let $\theta = (\theta_1, \dots, \theta_n)$ and $y = (y_1, \dots, y_n)$, $0 = y_0 \leq y_1 \leq \dots \leq y_n \leq 1$, and set $T_0 = 0$, $T_j = [Ty_j]$ for $j = 1, \dots, n$.

Theorem 1 is a consequence of the following lemma.

LEMMA 4.1. *Under the conditions of Theorem 1,*

$$L - \lim_{T \rightarrow \infty} \frac{1}{T^{1/\rho} L_\rho(T)} X^*(Ty) \stackrel{\Delta}{=} \mu^{-1/\rho} \Lambda_\rho(y).$$

PROOF. For any real number $0 \leq S \leq T$, define the number of renewals in the half-open interval $(S, T]$ by

$$K(S, T] = \begin{cases} K([0, T]) - K([0, S]) & \text{if } S \neq 0 \\ K(T) & \text{if } S = 0, \end{cases}$$

where $K(T) = K[0, T]$ as defined by relation (3.3) and where A/B denotes the set difference $A \cap B^c$. Then

$$(4.1a) \quad W^*(T_j) - W^*(T_{j-1}) = (T_j - T_{j-1})W_{K(T_{j-1})} I(K(T_{j-1}, T_j] = 0) \\ + [(S_{K(T_{j-1})} - T_{j-1})W_{K(T_{j-1})} + \sum_{k=K(T_{j-1})+1}^{K(T_j)-1} U_k W_k \\ + (T_j - S_{K(T_j)-1})W_{K(T_j)}] I(K(T_{j-1}, T_j] \geq 1),$$

$$(4.1b) \quad V^*(T_j) - V^*(T_{j-1}) = U_0 W_0 I(T_{j-1} < S_0 \leq T_j) + \sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k K_k I(K(T_{j-1}, T_j] \geq 1).$$

In either case $X = W$ or $X = V$, we write $X^*(T_j) - X^*(T_{j-1}) = A_{X,j}(T) + B_{X,j}(T)$, with $A_{X,j}(T)$ being the term with the sum on the right side of (4.1a) and (4.1b) respectively. We will show that there exist i.i.d. copies $\{\Lambda_\rho^{(j)}, 1 \leq j \leq n\}$ of Λ_ρ such that

$$(4.2) \quad L - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) A_{X,j}(T) = \mu^{-1/\rho} \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) (y_j - y_{j-1})^{1/\rho} \Lambda_\rho^{(j)},$$

$$(4.3) \quad L - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} B_{X,j}(T) = 0.$$

Combining (4.2) and (4.3) would then lead to

$$L - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) [X^*(T_j) - X^*(T_{j-1})] \\ = \mu^{-1/\rho} \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) (y_j - y_{j-1})^{1/\rho} \Lambda_\rho^{(j)},$$

and thus complete the proof of the lemma.

We first verify (4.2) when $X = W$. Since $K(T_j) \sim K(Ty_j) \sim \mu^{-1}Ty_j$, the joint limiting distribution of

$$A_{X,j}(T) = \left[\sum_{k=K(T_{j-1})+1}^{K(T_j)-1} U_k W_k \right] I(K(T_{j-1}, T_j] \geq 1)$$

is identical to that of

$$\left[\sum_{k=[\mu^{-1}Ty_{j-1}]+1}^{[\mu^{-1}Ty_j]} U_k W_k \right] I(K(T_{j-1}, T_j] \geq 1).$$

$P(K(T_{j-1}, T_j] \geq 1) \rightarrow 1$ yields $I(K(T_{j-1}, T_j] \geq 1) \rightarrow 1$ as $T \rightarrow \infty$ with probability one and, therefore, the joint distribution of $A_{X,j}(T)$ is identical to that of

$$(4.4) \quad A_j(T) = \sum_{k=[\mu^{-1}Ty_{j-1}]+1}^{[\mu^{-1}Ty_j]} U_k W_k.$$

Using Proposition 4.1 and the notation (2.1), we get

$$(4.5) \quad L - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} A_j(T) = \mu^{-1/\rho} (y_j - y_{j-1})^{1/\rho} \Lambda_\rho^{(j)},$$

where the $\Lambda_\rho^{(j)}$, $j = 1, 2, \dots$ are distributed as Λ_ρ in (3.2). They are also independent because the summation in (4.4) is over disjoint intervals. A similar argument holds when $X = V$.

Regarding (4.3), we note that when $X = W$,

$$\begin{aligned} B_{W,j}(T) &= (T_j - T_{j-1}) W_{K(T_{j-1})} I(K(T_{j-1}, T_j] = 0) + [(S_{K(T_{j-1})} - T_{j-1}) W_{K(T_j)} \\ &\quad + (T_j - S_{K(T_j)-1}) W_{K(T_j)}] I(K(T_{j-1}, T_j] \geq 1) \\ &=: B_{j,1}(T) + B_{j,2}(T) + B_{j,3}(T). \end{aligned}$$

$P - \lim_{T \rightarrow \infty} T^{-1/\rho} L_\rho(T)^{-1} B_{j,1}(T) = 0$ (convergence in probability) follows from Chebychev's inequality. Indeed, let $\gamma < \rho$. Then

$$\begin{aligned} P(T^{-1/\rho} L_\rho(T)^{-1} B_{j,1}(T) \geq x) &\leq [T^{-1/\rho} L_\rho(T)^{-1} x^{-1} (T_j - T_{j-1})]^\gamma E|W_0|^\gamma P(K(T_{j-1}, T_j] = 0) \\ &\sim x^{-\gamma} E|W_0|^\gamma \mu^{-1(\alpha-1)} (y_j - y_{j-1})^{\beta-\alpha+1} T^{(1-1/\rho)\gamma+1-\alpha} L_\rho^{-\gamma}(T) \end{aligned}$$

as $T \rightarrow \infty$, since $E|W_{K(T)}|^\gamma = E|W_0|^\gamma$, and, by (2.2)

$$P(K(T_{j-1}, T_j] = 0) = P(S_0 > T_j - T_{j-1}) \sim \mu^{-1(\alpha-1)} (T_j - T_{j-1})^{1-\alpha}.$$

But $\rho = \min(\alpha, \beta)$, $\alpha > 1$, and $\gamma < \rho$ yield $(1 - \frac{1}{\rho})\gamma + 1 - \alpha < 0$, so that we get the desired result.

In addition, with probability one,

$$\begin{aligned}
 |B_{j,2}(T)| + |B_{j,3}(T)| &\leq U_{K(T_{j-1})} |W_{K(T_{j-1})}| I(K(T_{j-1}, T_j] \geq 1) \\
 &\quad + U_{K(T_j)} |W_{K(T_j)}| I(K(T_{j-1}, T_j] \geq 1, K(T_j, \infty) \geq 1) \\
 &\leq U_{K(T_{j-1})} |W_{K(T_{j-1})}| + U_{K(T_j)} |W_{K(T_j)}|
 \end{aligned}$$

so that these terms, multiplied by $T^{-1/\rho} L_\rho(T)$, tend to zero in probability. Hence (4.3) holds for $X = W$, and similarly it holds for $X = V$. $\square$

PROOF of Theorem 1. Follows immediately from Lemma 1. $\square$

To prove Theorem 2 (case $X = V$) and Theorem 3, we need first to characterize the limiting process

$$Z_{V,T}(y) = L - \lim_{M \rightarrow \infty} \frac{1}{M^{1/\rho} L_\rho(M)} V^*(Ty, M), \quad 0 \leq y \leq 1,$$

for each $T = 1, 2, \dots$

LEMMA 4.2 ($X = V$). For each $T = 1, 2, \dots$

$$L - \lim_{M \rightarrow \infty} \frac{1}{M^{1/\rho} L_\rho(M)} V^*(Ty, M) \stackrel{\Delta}{=} Z_{V,T}(y)$$

where $Z_{V,T}(y)$, $0 \leq y \leq 1$, is a stable process with independent non-stationary increments, satisfying as $x \rightarrow \infty$:

(i) when $\rho = \beta$ ($\beta < \alpha$),

$$\begin{aligned}
 (4.6) \quad P\left(\sum_{j=1}^n \theta_j Z_{V,T}(y_j) \leq -x\right) &\sim \sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^\beta c^{-\sigma(\sum_{i=j}^n \theta_i)} \left\{ EU_0^\beta P(T_{j-1} < S_0 \leq T_j) \right. \\
 &\quad \left. + E \left[\sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k^\beta I(K(T_{j-1}, T_j] \geq 1) \right] \right\} x^{-\beta};
 \end{aligned}$$

$P\left(\sum_{j=1}^n \theta_j Z_{V,T}(y_j) \geq x\right)$ satisfies the same expression with $-\sigma$ replaced by σ .

(ii) when $\rho = \alpha$ ($\alpha < \beta$),

$$(4.7) \quad P\left(\sum_{j=1}^n \theta_j Z_{V,T}(y_j) \leq -x\right) \sim \left[\sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^\alpha E \left[W_0^{-\sigma(\sum_{i=j}^n \theta_i)} \right] P(T_{j-1} < S_0 \leq T_j) \right] x^{-\alpha};$$

with the other tail satisfying the same expression with $-\sigma$ replaced by σ .

PROOF. Using (4.1b), we write

$$\sum_{j=1}^n \theta_j V^*(T_j) = \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) [U_0 W_0 I(T_{j-1} < S_0 \leq T_j) + \sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k W_k I(K(T_{j-1}, T_j] \geq 1)].$$

(i) $\beta < \alpha$. Here W_0 controls the tail behavior of $U_0 W_0$. Conditioning on all of the U_k 's, using the independence of the W_k 's, letting $x \rightarrow \infty$, and finally unconditioning yield, as in Proposition 4.1,

$$P\left(\sum_{j=1}^n \theta_j V^*(T_j) \leq -x\right) \sim \left\{ E|A|^{\beta} c^{-\sigma(A)} E U_0^{\beta} + \sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\beta} c^{-\sigma\left(\sum_{i=j}^n \theta_i\right)} \cdot E \left[\sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k^{\beta} I(K(T_{j-1}, T_j] \geq 1) \right] \right\} x^{-\beta} g(x)$$

where $A = \sum_{j=1}^n \left(\sum_{i=j}^n \theta_i \right) I(T_{j-1} < S_0 \leq T_j)$. Since

$$E|A|^{\beta} c^{-\sigma(A)} = \sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\beta} c^{-\sigma\left(\sum_{i=j}^n \theta_i\right)} P(T_{j-1} < S_0 \leq T_j),$$

we obtain (4.6). The right probability tail satisfies the same expression with $-\sigma$ replaced by σ .

(ii) $\alpha < \beta$. U_0 controls the tail behavior of $U_0 W_0$, but W_k controls that of $U_k W_k$ when $K(T_{j-1}) \leq k \leq K(T_j) - 1$ and $K(T_{j-1}, T_j] \geq 1$ because then U_k is bounded by T_j , hence by T . Thus, using Proposition 4.1, as $x \rightarrow \infty$

$$\begin{aligned} P\left(\sum_{j=1}^n \theta_j V^*(T_j) \leq -x\right) &\sim \left[\sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\alpha} E \left(W_0^{-\sigma\left(\sum_{i=j}^n \theta_i\right)} \right)^{\alpha} P(T_{j-1} < S_0 \leq T_j) \right] x^{-\alpha} h(x) \\ &\quad + \sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\beta} c^{-\sigma\left(\sum_{i=j}^n \theta_i\right)} \\ &\quad \cdot E \left[\sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k^{\beta} I(K(T_{j-1}, T_j] \geq 1) \right] x^{-\beta} g(x) \\ &\sim \left[\sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\alpha} E \left(W_0^{-\sigma\left(\sum_{i=j}^n \theta_i\right)} \right)^{\alpha} P(T_{j-1} < S_0 \leq T_j) \right] x^{-\alpha} h(x), \end{aligned}$$

thereby verifying (4.6). The same expression holds for the right probability tail with σ replacing $-\sigma$.

Relations (4.6) and (4.7) show that in both cases (i) and (ii) the process $Z_{V,T}(y)$ is a stable process with independent increments. $\square$

To prove Theorem 2 ($X = V$) and Theorem 3, we must establish the exact growth rate of the right side of (4.6) and (4.7) as $T \rightarrow \infty$.

PROOF of Theorem 2. When $\beta < \alpha$ and $X = V$, the first term on the right side of (4.6) satisfies

$$(4.8) \quad \sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\beta_c} e^{-\sigma(\sum_{i=j}^n \theta_i)} E U_0^\beta P(T_{j-1} < S_0 \leq T_j) \sim \left| \sum_{i=1}^n \theta_i \right|^{\beta_c} e^{-\sigma(\sum_{i=1}^n \theta_i)} E U_0^{\beta(1-\mu^{-1})}$$

as $T \rightarrow \infty$, because $\lim_{T \rightarrow \infty} P(T_0 < S_0 \leq T_1) = P(S_0 > 0) = 1 - \mu^{-1}$ (see (2.2)) while

$\lim_{T \rightarrow \infty} P(T_{j-1} < S_0 \leq T_j) = 0$ for $2 \leq j \leq n$.

The second term on the right side of (4.6) is estimated as follows. For fixed j ,

$$\begin{aligned} & E \left[\sum_{k=K(T_{j-1})}^{K(T_j)-1} U_k^\beta I(K(T_{j-1}, T_j] \geq 1) \right] \\ &= E \left[\sum_{k=0}^{\infty} U_k^\beta I(K(T_{j-1}) \leq k \leq K(T_j) - 1, K(T_{j-1}, T_j] \geq 1) \right] \\ (4.9) \quad &= E \left[\sum_{k=0}^{\infty} U_k^\beta I(T_{j-1} < S_k \leq T_j) \right] \\ &= E \left[\sum_{k=0}^{\infty} U_k^\beta I(S_k \leq T_j) \right] - E \left[\sum_{k=0}^{\infty} U_k^\beta I(S_k \leq T_{j-1}) \right]. \end{aligned}$$

By Lemma 4.3 below, this is asymptotically proportional to

$$(4.10) \quad \mu^{-1} E U_0^\beta (y_j - y_{j-1}) T$$

as $T \rightarrow \infty$, dwarfing (4.8).

Lemma 4.2 provided the limit as $M \rightarrow \infty$. The above computation shows that if, in addition, one normalizes by $T^{1/\beta}$ and lets $T \rightarrow \infty$, then one obtains the Lévy stable process $\mu^{-1/\beta} \Lambda_\beta(y)$. This establishes Theorem 2 in the case $X = V$. $\square$

LEMMA 4.3. If $\beta < \alpha$, then as $T \rightarrow \infty$

$$(4.11) \quad E\left[\sum_{k=0}^{\infty} U_k^{\beta} I(S_k \leq T)\right] \sim \mu^{-1} EU_0^{\beta} T.$$

PROOF. Let $A(T) = E\left[\sum_{k=0}^{\infty} U_k^{\beta} I(S_k \leq T)\right]$. Then

$$A(T) = \sum_{k=0}^{\infty} \sum_{x=1}^T x^{\beta} P(U_k = x, S_k \leq T) = \sum_{x=1}^T x^{\beta} P(U_k = x) \sum_{k=1}^{\infty} P(S_{k-1} \leq T - x)$$

where we used the relation $\{U_k = x, S_k \leq T\} = \{U_k = x, S_{k-1} \leq T - x\}$ and the independence of U_k and S_k . Now,

$$\sum_{k=1}^{\infty} P(S_{k-1} \leq T - x) = \sum_{k=1}^{\infty} P(K(T - x) \geq k) = EK(T - x) = \mu^{-1}(T - x),$$

since $\{S_k, k = 0, 1, \dots\}$ is stationary ([R], Thm 3.15(i), p. 48). Thus,

$$(4.12) \quad A(T) = \mu^{-1} T \sum_{x=1}^T x^{\beta} P(U_0 = x) - \mu^{-1} \sum_{x=1}^T x^{\beta+1} P(U_0 = x).$$

The first term is asymptotic to $\mu^{-1}(EU_0)T$ as $T \rightarrow \infty$. Since $P(U_0 = x) \sim \alpha x^{-\alpha-1} h(x)$ as $x \rightarrow \infty$, the second term satisfies, as $T \rightarrow \infty$,

$$\mu^{-1} \sum_{x=1}^T x^{\beta+1} P(U_0 = x) \sim \begin{cases} \mu^{-1} EU_0^{\beta+1} & \text{if } EU_0^{\beta+1} < \infty \\ \mu^{-1} \alpha(\beta-\alpha+1)^{-1} T^{\beta-\alpha+1} h(T) & \text{if } EU_0^{\beta+1} = \infty \end{cases}$$

and hence is $o(T)$ because $\beta < \alpha$. It is negligible compared to the first term. $\square$

PROOF of Theorem 3. Here $\alpha < \beta$ and $X = V$. In this case, relation (4.7) of Lemma 4.2 characterizes the limit as $M \rightarrow \infty$. To get the limit as $T \rightarrow \infty$, note that

$$\lim_{T \rightarrow \infty} P(T_{j-1} < S_0 \leq T_j) = 0 \text{ for } j = 2, \dots, n \text{ but, if } j = 1, \lim_{T \rightarrow \infty} P(T_{j-1} < S_0 \leq T_j) = P(S_0 > 0) = 1 - \mu^{-1}. \text{ Thus, as } T \rightarrow \infty,$$

$$\sum_{j=1}^n \left| \sum_{i=j}^n \theta_i \right|^{\alpha} E \left(W_0^{-\sigma \left(\sum_{i=j}^n \theta_i \right)} \right)^{\alpha} P(T_{j-1} < S_0 \leq T_j) \sim \left| \sum_{i=1}^n \theta_i \right|^{\alpha} E \left(W_0^{-\sigma \left(\sum_{i=1}^n \theta_i \right)} \right)^{\alpha} (1 - \mu^{-1}).$$

The limiting process $\Lambda_{\alpha,+}(y)$ is thus constant for $y \neq 0$ and equal to the stable random variable characterized by (3.6). At $y = 0$, we have $\Lambda_{\alpha,+}(0) = 0$ because $V^*(0) = 0$. $\square$

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