

QUATERNIONIC FATOU-JULIA SETS

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Résumé

Notre travail porte sur les systèmes dynamiques engendrés par les transformations quadratiques agissant sur les quaternions. L'étude de ces systèmes procède par analogie avec la théorie classique des itérées de fonctions rationnelles dans le plan complexe, telle qu'élaborée par Fatou et Julia. Les "dragons" de Fatou-Julia dont on est familier depuis les travaux de Mandelbrot admettent des versions dans l'espace des quaternions. On peut traiter ces dragons par ordinateur comme des objets géométriques et les représenter par illumination artificielle et selon des ombres. On examinera les symétries de ces objets et certaines de leurs propriétés topologiques comme la connexité. Nous sommes près d'un travail apparenté de Mandelbrot et Norton. Nous proposerons enfin un ensemble de Mandelbrot pour les quaternions.

Summary

We outline our work on the dynamics of quadratic self-maps of the quaternions. This proceeds by analogy with the classical study of iterated rational maps of the complex plane, due to Fatou and Julia. The Fatou-Julia "dragons", now fami-

liar through the work of Mandelbrot, have quaternionic variants that may be computer-modelled as three-dimensional geometric objects, and displayed via (artificial) light and shadow. The symmetries and connectivity of these objects are examined. We touch upon related work of Mandelbrot and Norton, and propose a quaternionic Mandelbrot set.

Acknowledgement

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1. Introduction

Benoît Mandelbrot's idea of using computer graphics to display sets in the complex plane that correspond to classical Fatou-Julia theory has been so successful that his "dragons" are now familiar objects (see [5], for example). His analysis of the corresponding parameter space, leading to what we now call the Mandelbrot set, has also created a sensation (see A.K. Dewdney [2], for one presentation). For our purposes it will be enough to consider the quadratic maps f_μ defined by $f_\mu(z) = z^2 + \mu$. For the moment, let us consider μ to be a fixed complex number, and z to be complex variable. The classical Julia set corresponding to the map f_μ may be described as the boundary of the following "Fatou-Julia" set associated with μ :

$$FJ(\mu) = \{z \in \mathbb{C} : \text{the sequence } \{f_\mu^n(z)\} \text{ is bounded}\}.$$

Here we use the convention that f^n denotes the n -fold iterate of a map f under composition. For appropriate choices of parameter μ , the sets $FJ(\mu)$ are fantastic in shape, often described as "dragons". The boundary of $FJ(\mu)$ (the Julia set) is generally a fractal, meaning, in this context, that it has Hausdorff dimension greater than one. The Mandelbrot set M is the set of complex parameters μ such that $0 \in FJ(\mu)$; this last condition turns out to be equivalent to the connectedness of $FJ(\mu)$. For a modern account of these features of complex dynamics, see P. Blanchard [1].

Our original motivation for extending these ideas to the quaternionic setting was to see what 3-D dragons might look like. Some dragon-like beasts do occur (see Figures 1A and 1B, for example), but there are other interesting phenomena as well. Since the quaternions $\mathbb{H}$ provide a ready-made higher-dimensional analogue of the complex numbers, it is natural to consider, given a quaternion μ as parameter, the set

$$\text{QFJ}(\mu) = \{q \in \mathbb{H} : \text{the sequence } \{f_\mu^n(q)\} \text{ is bounded}\}.$$

Here, of course, $f_\mu(q)$ is interpreted as $q^2 + \mu$, where q^2 is computed via quaternionic multiplication. Since $\mathbb{H} \approx \mathbb{R}^4$, $\text{QFJ}(\mu)$ is a four-dimensional object. Appropriate sections may nevertheless be displayed on the video screen or in a printed figure as three-dimensional solids whose shapes are revealed by computing artificial light and shadow and other modelling effects. It appears that Alan Norton was the first to investigate this procedure; brief accounts of his remarkable results, and of his collaboration with Mandelbrot may be found in [5; especially the update appended to later printings], [6], [7], and [8]. We understand that a more thorough presentation of this work has been prepared by Mandelbrot and Norton and will appear in due course.

This article is closely related to our lecture at the Fractal Geometry Workshop, Centre de recherches mathématiques, Université de Montréal, May 1986. We first discuss some examples of QFJ-displays, and the techniques used to produce them. Next we discuss certain symmetries that simplify such displays. There follows a preliminary study of the internal structure of QFJ-sets, with emphasis on their connectivity, and the role of the central "ring system". Finally we discuss possible quaternionic versions of the Mandelbrot set. An account of some of our earlier work on QFJ-sets appeared as [4].

2. Displaying QFJ-sets

Elementary estimates using the properties of the modulus $|q|$ of a quaternion q (which is also the Euclidean length of q as a vector in $\mathbb{R}^4$), show that

$R(\mu) = 0.5 + (|\mu| + 0.25)^{\frac{1}{2}}$ has the following property: if $|q| > R(\mu)$ then $|q| < |f_\mu(q)| < |f_\mu^2(q)| < \dots$ and this sequence tends to ∞ . It follows that the complement of $QFJ(\mu)$ in H is the basin of attraction $A(\infty)$ of the point at infinity " ∞ " in R^4 . Moreover, the sequence of open sets $O_n = \{q \in H: |f_\mu^n(q)| > R(\mu)\}$ increases to fill up $A(\infty)$.

Given μ , q , and n , it is clearly a straightforward matter to check whether $q \in O_n$, and this is the basis of our computer generated images of QFJ-sets. The most important steps are the following:

- (1) choose a three-dimensional hyperplane HP in H ;
- (2) choose a sufficiently fine (finite) grid G of points in the central cube of HP having semiaxes of length $R(\mu)$;
- (3) choose an integer N so large that O_N is a good approximation to $A(\infty)$, relative to the resolution afforded by the grid G ;
- (4) for each $q_0 \in G$, compute successively the iterates $q_n = f_\mu^n(q_0)$, checking to see whether $|q_n|$ exceeds the "escape radius" $R(\mu)$;
- (5) continue if necessary until $n = N$, recognizing that $|q_N| \leq R(\mu)$ means that $q_0 \in C_N$, the set of points of G in the complement of O_N ;
- (6) display C_N as a good approximation to $QFJ(\mu) \cap HP$.

We shall presently say more about the "display" step (6), where one is faced with the problem of creating two-dimensional images that convey something of the shape of the three-dimensional object C_N . In fact, the display procedures interact with the earlier steps and allow us to reduce the number of computations considerably from what would appear to be necessary at first glance. We shall not attempt to describe in any detail the computer programs actually used to carry out this procedure. It will be clear that some compromise must be made between the degree of approximation to $QFJ(\mu)$ desired and the time and money available to carry out the computer runs. The data used to produce a typical figure in this article would initially result from mainframe computations with $N = 80$ and a grid G obtained from dividing each side of the display cube into 480 equal parts. This implies 480^2 pixels on the video display (we used IBM Professional Graphics gear), but

we shall see later that not all of the 480^3 grid points need to be examined. Nevertheless, the volume and intricacy of our computations is considerable and we are fortunate to have the help of Jean-Pierre Schoch in keeping the machines under control.

Figure 2 shows a sample view of a QFJ-set obtained by the steps just outlined. Here $\mu = (-0.5, 0.5, 0, 0)$, or $-0.5 + 0.5i$ (we shall use the traditional i, j , and k to denote the "imaginary" quaternionic units; thus $1, i, j, k$ may also be regarded as the standard basis of unit vectors in $\mathbb{R}^4$). The hyperplane HP has here been chosen as the span of $1, i$, and j , and we are viewing $QFJ(\mu) \cap HP$ (or rather its approximation C_{80}) from a "generic" vantage point in the direction of $1 + i + j$. Some of the shading effects are the result of simulated "sunlight" coming from the right (more about this later). Many of the same remarks apply to Figure 3, where $\mu = 0.3795 - 0.3072i$. Here however we are viewing the "3-D slice" from the direction of the j -axis, and we have removed the part of the body with positive j -component, revealing the cross-section obtained by intersecting $QFJ(\mu)$ with the plane spanned by 1 and i . This plane is just a copy of the complex plane, and since it is a subalgebra of H and since μ is also complex in this example it is clear that the cross-section is (an approximation to) the classical $FJ(\mu)$. In fact this dragon-like section $FJ(0.3795 - 0.3072i)$ is a close relative of the well-known dragon adorning the dust-cover of [5].

Figure 4 displays a more "generic" 3-D slice of $QFJ(0.3795 - 0.3072i)$, obtained by taking HP as the subspace orthogonal to $1 + i + j + k$. The wheel-like structure is not really a solid of revolution, although it may seem to be one at first glance. The section of this same body revealed in Figure 5 makes the asymmetry clear and shows that "generic" planar sections of $QFJ(0.3795 - 0.3072i)$ retain the dragon-like quality of $FJ(0.3795 - 0.3072i)$.

In all these figures, the display techniques guide the computations. Once HP and the viewing direction are specified, the programs run "visual probes" into the display cube along each of the 480^2 grid lines parallel to the viewing direction.

These may be indexed by integer pairs (α, β) , where $1 \leq \alpha, \beta \leq 480$, and our computations ultimately produce an array PIX in which $PIX(\alpha, \beta)$ determines the color to be displayed as the video screen's (α, β) -th pixel. A quaternion q_0 located along the (α, β) -th grid line is successively mapped under f to $q_1, q_2, \dots$, and may escape at the n -th step ($|q_n| > R(\mu)$). If $n < N$ we must advance along the grid line a certain distance (determined by estimating the Lipschitz constant of f^n) before there is the possibility of encountering C_N . Because of this we need not deal with every grid point along the line, and some computation is avoided.

If C_N is encountered along the (α, β) -th grid line, $PIX(\alpha, \beta)$ initially records the depth to the nearest point in C_N (measured, let us say, from the front of the display cube). Later, the procedure computes relative depths to assign deeper shades of color to the more distant points on the body C_N . This is one way in which the 3-D structure of C_N is conveyed. We may think of this technique as illumination of C_N by means of a flashlight (weak light source) held at the observer's position. Distant parts of C_N are more dimly lit. In a second stage of the procedure, we determine whether each point of C_N found in the first stage is or is not illuminated by the sun (strong light source). The sun is assumed to shine from a direction to the right of the screen and somewhat to the front (in a ratio of 2 units to the right for each unit toward the observer). Starting from a point on C_N the program traces out the path in this direction (i.e. towards the sun) to see whether C_N is encountered again; in this case the original point is in the shade, and in the final version of PIX the corresponding $PIX(\alpha, \beta)$ assigns bluish hues to the pixel. If on the other hand, the sun-seeking path gets beyond the escape radius $R(\mu)$, it is clear that no obstructing part of C_N is encountered and the original point is deemed to be sunlit; ultimately $PIX(\alpha, \beta)$ assigns it a yellowish hue. The resulting pattern of light and shade provides further visual clues to the 3-D structure of C_N .

We remark that in many cases the depth information stored in PIX is all that is needed to check that a sun-seeking path lies entirely in front of C_N , so that the point of origin is sunlit. We shall see, however, that there may be tunnels

through C_N , so that further computations are needed at times to check whether a point is illuminated by sunlight passing through a tunnel (see Figure 8) or through some other hidden channel in C_N .

Some of our figures (e.g. Figure 1B) display, along with the features outlined above, an "aura" in the space surrounding C_N . This aura provides clues to the proximity of C_N : the outer aura (relatively far from C_N , but within the escape radius $R(\mu)$) is dark, while the inner aura (close to C_N) is more intensely colored. This aura is determined by keeping track, as we proceed along a grid line (α, β) that misses C_N , of the maximum number n of iterations needed to achieve escape ($|q_n| > R(\mu)$). For such a grid line, this maximum is less than N , but the larger its value the closer the (α, β) -th grid line gets to C_N . This value is therefore used to compute the aura-intensity of the (α, β) -th pixel.

3. Symmetries

Since $f_\mu(-q) = f_\mu(q)$, each $QFJ(\mu)$ is symmetric under reflection through the origin in $\mathbb{R}^4$. This symmetry through the origin will be seen in C_N whenever HP is a (three-dimensional) subspace, i.e., when the slicing hyperplane passes through the origin.

There is another, less obvious, sort of symmetry. We may write any quaternion q in the form $x + tU$, where x and t are real and U is a pure quaternion of unit length, i.e., U is a unit vector in the span of i, j , and k . If q' is similarly expressed as $x' + t'U'$, the quaternionic product may be computed as follows:

$$qq' = (xx' - tt'U \cdot U') + xt'U' + x'tU + tt'U \times U'.$$

Here $U \cdot U'$ and $U \times U'$ denote the dot product and cross product of U and U' , considered as 3-vectors. In this spirit, let us write the parameter μ as $a + bU$ and express an arbitrary quaternion q as $x + yU + zV$, where x, y, z are real and U, V are unit pure quaternions with V orthogonal to U . Since

$U \times V + V \times U = 0$ it is easy to check, using the formula above, that

$$f_{\mu}(q) = (x^2 - y^2 - z^2 + a) + (2xy + b)U + 2xzV.$$

Thus $q \in \text{span}(1, U, V)$ implies that $f_{\mu}(q) \in \text{span}(1, U, V)$, and the orbit of q is determined by the initial values of x , y , and z and the three-dimensional quadratic map F , depending only on a and b , defined by

$$F(x, y, z) = (x^2 - y^2 - z^2 + a, 2xy + b, 2xz).$$

The condition $q \in \text{QFJ}(\mu)$ is therefore a condition on x , y , and the distance $|z|$ from $\text{span}(1, U)$. This plane is simply the plane determined by the reals and the parameter μ . Clearly this plane is a plane of symmetry for any slice of $\text{QFJ}(\mu)$ by a hyperplane HP that contains the plane.

Consequently we may as well orient our QFJ -set so that $U = i$, i.e., choose $\mu \in \mathbb{C}$, and consider the hyperplane $HP = \text{span}(1, i, j)$. This convention is followed in most of this article. Let us call the corresponding slice $\text{QFJ}(\mu) \cap HP$ the standard slice SS . Once SS is known the complete QFJ -set may be reconstructed as

$$\text{QFJ}(\mu) = \{x + yi + rj + sk : (x, y, (r^2 + s^2)^{\frac{1}{2}}) \in SS\}.$$

In this sense the $\text{QFJ}(\mu)$ constitute, except for rotations in $\mathbb{H}$, simply a family of 3-D objects parametrized by $\mathbb{C}$ and obtained by "fattening" in a specific way each classical $\text{FJ}(\mu)$.

Note that $q = (0, y, z) \in SS$ exactly when $f_{\mu}(q) \in SS$; also

$$f_{\mu}(q) = -y^2 - z^2 + \mu = f_{\mu}(i(y^2 + z^2)^{\frac{1}{2}}).$$

Thus each SS has a "ring system" $RS = SS \cap \text{span}(i, j)$, which may be obtained from the classical $\text{FJ}(\mu)$ by rotating $\text{FJ}(\mu) \cap i\mathbb{R}$ about the real axis. Figure 6 shows the ring system RS for $\text{QFJ}(0.3795 - 0.3072i)$ (featured also in Figures 3, 4, and 5). We shall see that the nature of the ring system affects the connectivity of SS (and $\text{QFJ}(\mu)$).

4. Connectivity

It is not hard to see that the complement of $FJ(\mu)$ in C is connected. Let us use, for the moment, the notations $A(\infty)$ and O_n in this classical case also. Since the open sets O_n increase to $A(\infty)$, it is enough to show that each O_n is connected; but O_n includes the neighborhood O_0 of ∞ so that O_n can only be disconnected if it contains a bounded component B . Now $O_n = \{z \in \mathbb{C}: |f_\mu^n(z)| > R(\mu)\}$ so that $|f_\mu^n(z)| = R(\mu)$ on the boundary of B and the maximum of $|f_\mu^n(z)|$ over B must be attained at an interior point. This is not possible, however, since f_μ^n is analytic and therefore an open mapping so that $|f_\mu^n|$ cannot have any local maxima. Hence no such B exists.

We wish to argue that $A(\infty)$, the complement of $QFJ(\mu)$, is connected in the quaternionic setting also. In view of the symmetries discussed in the previous section, it is enough to show that CS , the complement of SS in $\mathbb{R}^3$, is connected. Clearly CS is the union of the increasing sequence of open sets U_n in $\mathbb{R}^3$ described as follows:

$$U_n = O_n \cap \text{span}(1, i, j) = \{(x, y, z): |F^n(x, y, z)| > R(\mu)\}.$$

These relations may be clearer if we confess to several notational shortcuts: $\text{span}(1, i, j)$ is identified with $\mathbb{R}^3$, $x + yi + zj$ with (x, y, z) , and $|p|$ with the Euclidean length of p in $\mathbb{R}^3$. Again let B be a hypothetical bounded component of U_n ; since F^n is not an open mapping, we cannot proceed quite as we did in the classical case. It is easy to check, however, that the Jacobian of F is singular only on the plane $\{x = 0\}$. Hence, if p_0 is the point of B at which the maximum of $|F^n|$ over B is attained, we must have $p_m = F^m(p_0) \in \{x = 0\}$ for some $m < n$ (otherwise F^n would be an open mapping at p_0 , contradicting the assumed extremal property of p_0). We may assume that m is minimal with the property that $F^m(B)$ meets $\{x = 0\}$, for otherwise we need only replace p_0 by some other point in B . Consequently, $(F^m)'$ is nonsingular on B , so that F^m is open at each point $p \in B$. Since $p_0 \notin SS$, p_m must lie in one of the gaps in the ring system RS (or entirely outside it). Following around this gap we arrive at a point in the

complement of $FJ(\mu)$ in C , and using the classical connectivity result for this set we can join the point to ∞ . We thus have a path from p_m to ∞ lying entirely in CS . In fact our discussion of the classical connectivity result implies that, since $p_m \in U_{n-m}$, we may assume that this path lies entirely in U_{n-m} , i.e., we have a continuous map Γ from $[0, \infty)$ into U_{n-m} such that $\Gamma(0) = p_m$, and $|\Gamma(t)|$ tends to ∞ with t . Now B is bounded, so $F^m(B)$ is also bounded, and we may consider $T = \sup\{t: \Gamma(t) \in F^m(B)\}$. By Bolzano-Weierstrass, we can choose $b_h \in B$ such that $p = \lim\{b_h\}$ exists and $F^m(b_h) = \Gamma(t_h)$ with t_h approaching T . But then $F^m(p) = \Gamma(t) \in U_{n-m}$, so that $p \in B$; since F^m is open at p , this would imply that $F^m(B)$ includes some points $\Gamma(t)$ for $t > T$, a contradiction. We have established the following.

PROPOSITION 4.1. *The complement $A(\infty)$ of $QFJ(\mu)$ in H ($= \mathbb{R}^4$) is connected. Similarly, the complement of the special slice SS of $QFJ(\mu)$ is connected in $\mathbb{R}^3$.*

COROLLARY 4.2. *The special slice SS of $QFJ(\mu)$ includes no holes, but it may have tunnels (i.e., it may be multiply connected). This is certainly the case when the classical $FJ(\mu)$ meets the imaginary axis nontrivially (i.e., intersection is more than a single interval).*

PROOF of Corollary. The ring system RS introduced in Section 3 will be nontrivial in the case. Any point p in a gap of the ring system may be connected by a path Γ to ∞ . It may also be connected to $-p$ within the ring system, and $-p$ is connected by $-\Gamma$ to ∞ in such a way that the combined loop is linked with any ring of RS lying outside p .

Q.E.D.

The QFJ -sets for $\mu = -0.5 + 0.5i$ and $\mu = 0.3795 - 0.3072i$ displayed in earlier figures both have nontrivial ring systems. Figures 7, 8, and 9 examine the tunnel effect that must accompany such ring systems.

5. Mandelbrot sets

For the classical FJ-sets the Mandelbrot set may be described as the set M of parameter values μ such that $FJ(\mu)$ is connected. Remarkably, this condition is equivalent to the easily checked relation $0 \in FJ(\mu)$, i.e., $\{|f_\mu^n(0)|\}$ bounded. One fascinating aspect of the structure of M is that it serves as something of an atlas of FJ-sets. For example, the "island molecules" of M are linked together by a fine webbing that mimics the pattern seen in $FJ(\mu)$ when μ is chosen appropriately from an island molecule. See Mandelbrot's papers in [3; pp. 213-253] for terminology and many results concerning the Mandelbrot set. Figure 1 features the SS of $QFJ(\mu)$ for μ from an island molecule; one desideratum for a quaternionic Mandelbrot set might be that its structure reflect that of such QFJ-sets.

Although it is tempting to consider $TQM = \{\mu \in H: 0 \in QFJ(\mu)\}$ in this light, the resulting set is too closely linked to the classical M . If we write $\mu = a + bU$, where U is a unit pure quaternion, the orbit $\{f_\mu^n(0)\}$ entails computations in $\text{span}(1, U)$ that are isomorphic to those for $a + bi$ in C . In fact, $a + bU \in TQM$ exactly when $a + bi \in M$. Thus TQM is merely a sort of rotation of M into R^4 .

Roughly speaking, we may explain the fact that M tends to look locally like the FJ-sets obtained from μ in a certain locality by observing that the orbit $\{f_{\mu'}^n(\mu')\}$ (which is just the orbit of 0 with the first element missing) will be close to the orbit $\{f_\mu^n(\mu')\}$ if μ' is close to μ . Boundedness of one orbit will tend to go along with boundedness of the other. This suggests the following definition for a quaternionic Mandelbrot set that might serve as an atlas of QFJ-sets (or rather of the corresponding SS):

$$QM = \{q = x + yi + zj: \{f_{x+iy}^n(q)\} \text{ is bounded}\}.$$

Figures 10, 11, and 12 show various views of QM . Of course, the section $\{z = 0\}$ is just a copy of M . Note that the island molecules appear to be elongated in the j -direction in the style of the QFJ-set in Figure 1A.

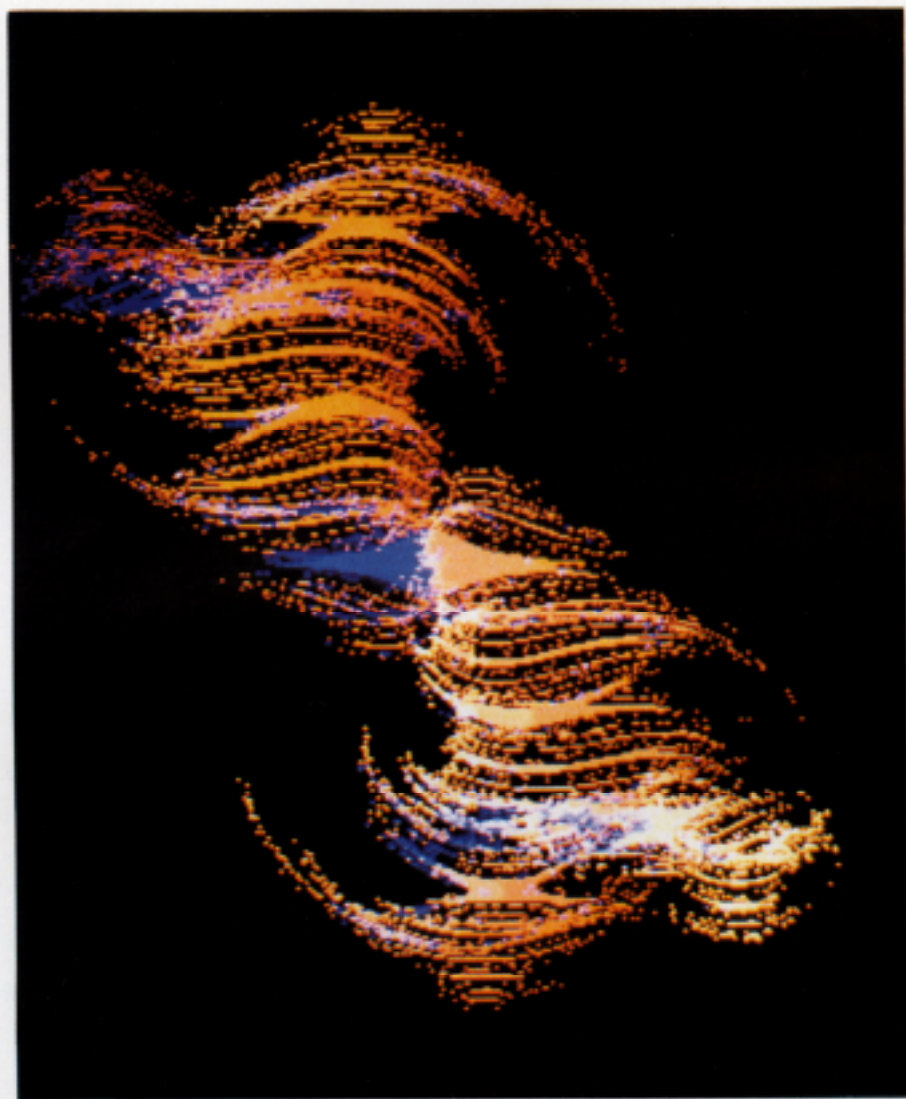


Figure 1A: A view of the part of QFJ $(-.157+1.032i)$ lying in the three-dimensional subspace (of the quaternions) spanned by 1 , i , and j . As explained in more detail in the text, $\text{QFJ}(\mu)$ is the set of quaternions q that are stable under the map $f_\mu(q)=q^2+\mu$ in the sense that their forward orbits are bounded. Here we are viewing the 3-D section of $\text{QFJ}(\mu)$ from the direction of $1+j$, with simulated illumination from a weak source ("flashlight") at our position and a strong source ("sun") at the right.

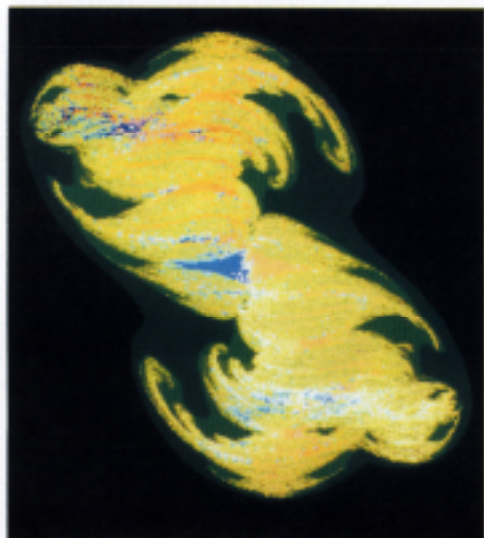


Figure 1B: $QFJ(-.157 + 1.032i)$ as in figure 1A, but with the addition of "aura"; the brightness of the green aura varies from point to point outside $QFJ(\mu)$, increasing as more iterations are required to attain the escape radius $R(\mu)$ (which verifies that the point in question does lie outside $QFJ(\mu)$).



Figure 2: $QFJ(-.5 + .5i)$ seen from the direction of $1 + i + j$ within the span of $1, i$, and j .

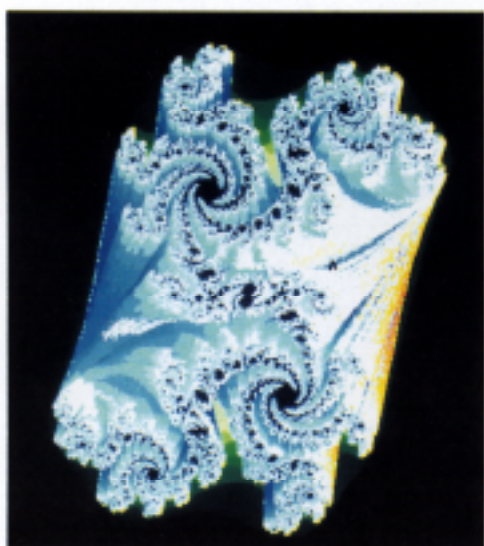


Figure 3: $QFJ(.3795 - .3072i)$ seen from the direction of j in the span of $1, i$, and j . Aura is displayed around the object, and indicates that very many iterations are required to scoop out some of the deep folds in $QFJ(.3795 - .3072i)$. Here the front of the object is cut away to reveal the cross-section, shown in black, with the complex plane. This is a version of one of the more famous Mandelbrot-Fatou-Julia dragons.

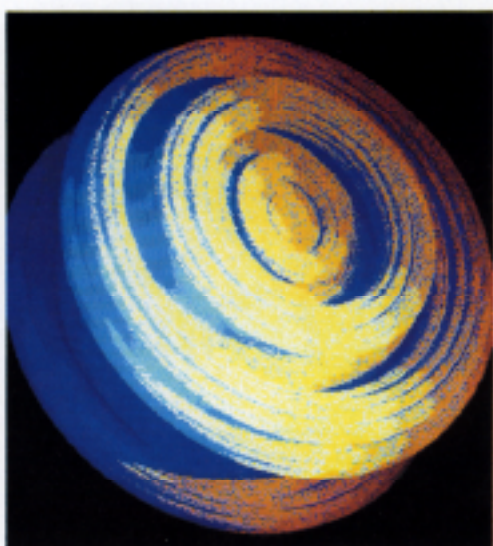


Figure 4: The 3-D slice of $QFJ(.3795 - .3072i)$ in the subspace of R^4 ($=H$) orthogonal to $(1, 1, 1, 1)$ ($=1 + i + j + k$).

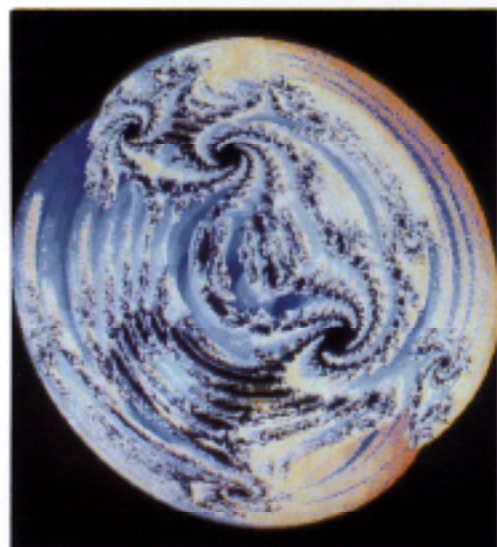


Figure 5: A cut-away view of the "wheel" not really a solid of revolution) $QFJ(.3795 - .3072i)$ as in figure 4. The rich internal structure is revealed, and in particular we see the reappearance (compare figure 3) of the dragon-like cross-section, shown in black.

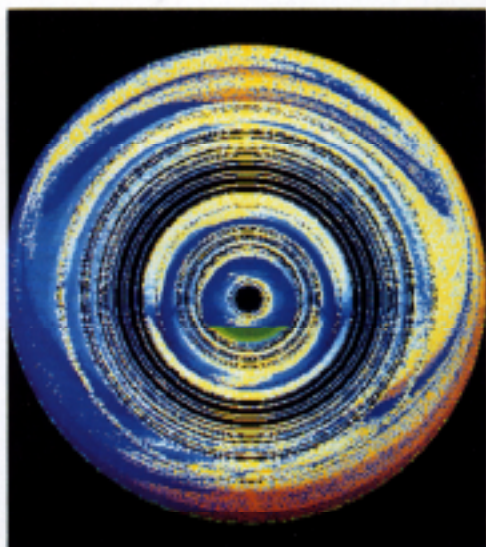


Figure 6: The complicated ring system RS of $QFJ(.3795 - .3072i)$. This induces the formation of tunnels through the standard slice SS of $QFJ(.3795 - .3072i)$.

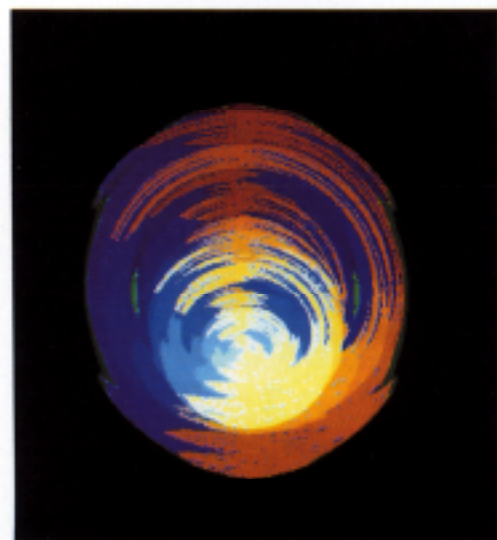


Figure 7: $QFJ(-.5 + .5i)$, or rather its standard slice SS, is viewed from the direction of 1 (real axis). The bright inner aura seen through the slits in the object indicates that the tunnels predicted by the ring system of this object are straight enough to see through.

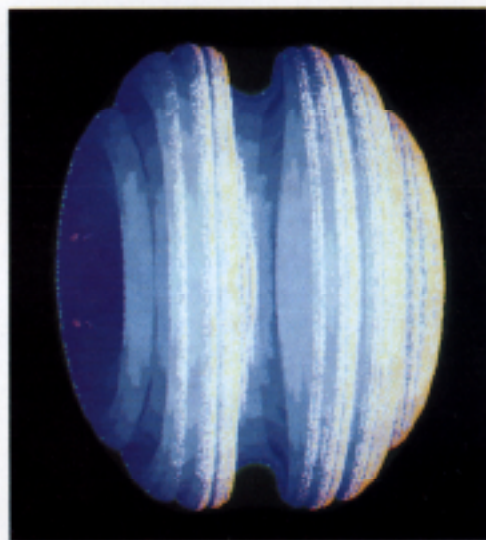


Figure 8: $QFJ(.3795 - .3072i)$ seen from the direction of i . The flecks of light seen within the shadowed region at the left reveal that there are tunnels in this object that are straight enough to pass some light.

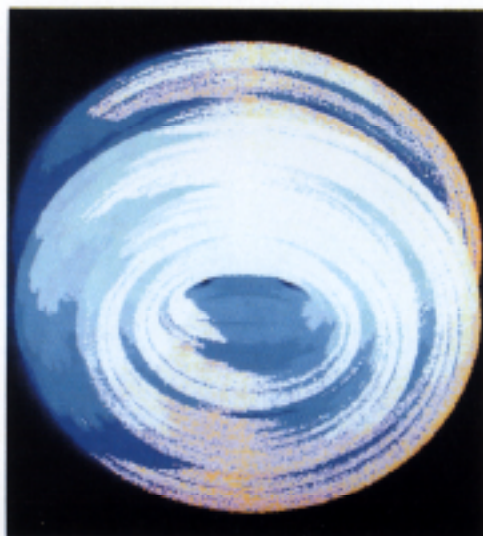


Figure 9: Another view of the SS of QFJ(.3795 - .3072i), exploiting the view through the object that is guaranteed by the evidence of figure 8.

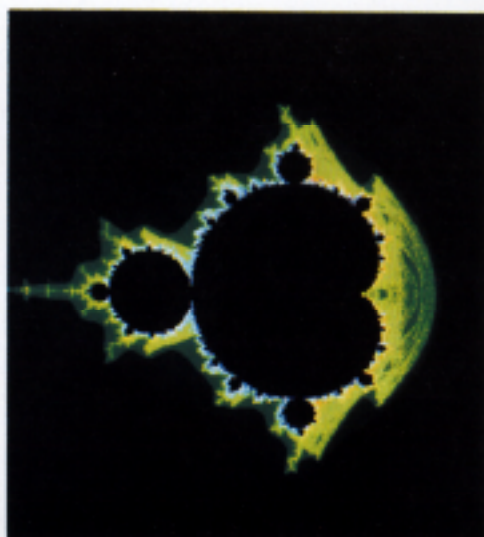


Figure 10: View of QM (see section 5 of the text) showing the cross-section (black) with the complex plane, which is a copy of the usual Mandelbrot set M . The aura (shades of green) is computed in the same spirit as for QFJ-sets.

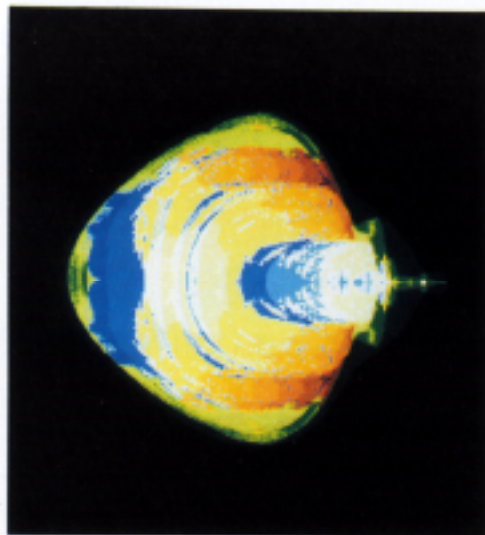


Figure 11: Another view of QM, from the direction of $-1 + j$. Note that the "island molecules" appear to be elongated in the manner of the related QFJ-set shown in figures 1A and 1B.

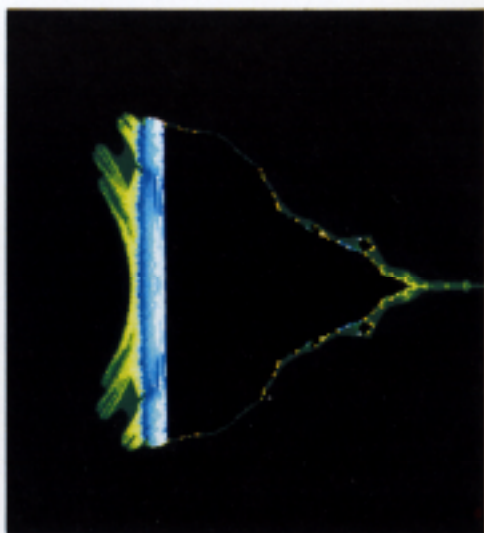


Figure 12: A section of QM (by the plane spanned by 1 and j) seen from the direction of i .

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