

A GENERALIZATION OF FENCHEL'S DUALITY THEOREM

Valise Postolică

Summary

The author studies a generalization of Fenchel's duality theorem to develop the study of Pareto optimization problems with objective maps multifunctions. This generalization is based on the concept of nuclear cone defined and studied by Isac.

Résumé

L'auteur étudie une généralisation du théorème de dualité de Fenchel pour développer l'étude de l'optimum de Pareto pour les problèmes d'optimisation avec les applications objectives multivalentes. Cette généralisation est basée sur le concept de cône nucléaire défini et étudié par Isac.

It is well known the fundamental importance of the Fenchel duality theorem in convex optimization. An extension of this theorem is given in [5] (Theorem 4.1) which considers duality in vectorial programming, using the concept of conjugate map for vector-valued functions.

Recently, we introduced in [4] the notions of vectorial subdifferential and vectorial conjugate for multifunctions defined on a real linear space and taking values in a real ordered linear space and we investigated their immediate properties

with the aim of extending a duality for vectorial optimization programs in which the objective maps are multifunctions, generalizing basic results of [5] especially for the case when the objective multifunctions take values in a quasicomplete locally convex space ordered by a closed and nuclear (supernormal) cone.

We note that the concept of nuclear cone was defined in [1] and it is very important in the theory of Pareto optimum. Others applications of this class of convex cones may be found in [2], [3].

The purpose of the present work is to continue and to complete our paper [4] with a generalization of Theorem 4.1 of [5], using the same notations.

Let $(Z, \leq)$ be a real linear space ordered by a convex cone Z_+ . We add to Z a smallest element denoted by $-\infty$ and a largest element denoted by $+\infty$ respectively, we consider $\bar{Z} = Z \cup \{-\infty, +\infty\}$ and we extend the addition and the scalar multiplication of Z to $\bar{Z}$ using the usual calculation conventions. Throughout the paper we denote by $\emptyset$ the empty set. If $A \neq \emptyset$, $\{+\infty\}$ is an arbitrary subset of $\bar{Z}$ and $p \in \bar{Z}$, then

DEFINITION 1. We shall say that $p \in \text{INF}(A)$ iff there exists no $a \in A$ such that $a < p$.

DEFINITION 2. We say that $p \in \text{INF}_1(A)$ iff $p \in \text{INF}(A)$ and for every $p' \in Z$ with $p < p'$, there exists $a \in A$ such that $a < p'$.

If $A = \emptyset$ or $A = \{+\infty\}$, then we consider $\text{INF}(A) = \bar{Z}$. In a similar manner we define the sets $\text{SUP}(A)$ and $\text{SUP}_1(A)$.

DEFINITION 3. We shall say that $p \in \text{SUP}(A)$ if there exists no $a \in A$ with $p < a$.

DEFINITION 4. We say that $p \in \text{SUP}_1(A)$ iff $p \in \text{SUP}(A)$ and for every $p' \in Z$ with $p' < p$, there exists $a \in A$ such that $p' < a$.

In fact, we have $\text{SUP}(A) = -\text{INF}(-A)$ and $\text{SUP}_1(A) = -\text{INF}_1(-A)$.

REMARK. If we denote $Z'_+ = Z_+ \setminus \{0\}$, $\text{MIN}(A) = A \cap \text{INF}(A) = A \cap \text{INF}_1(A)$ and $\text{MAX}(A) = A \cap \text{SUP}(A) = A \cap \text{SUP}_1(A)$, then the following conditions are satisfied:

- (i) $\text{INF}_1(A) = \text{INF}_1(A + Z_+)$;
- (ii) $\text{SUP}_1(A) = \text{SUP}_1(A - Z_+)$;
- (iii) $\text{MIN}[\text{MIN}(A)] = \text{MIN}(A)$;
- (iv) $\text{MAX}[\text{MAX}(A)] = \text{MAX}(A)$;
- (v) $\text{MIN}(A) = A \cap \text{MAX}[Z \setminus (A + Z'_+)] = A \cap [Z \setminus (A + Z'_+)]$;
- (vi) $\text{MAX}(A) = A \cap \text{MIN}[Z \setminus (A - Z'_+)] = A \cap [Z \setminus (A - Z'_+)]$;

for every nonempty set $A \subseteq Z$.

Before proving our main result, we shall make some preliminary considerations. Following [4], if X and Y are two real linear spaces, $f: D(f) \subseteq X \rightarrow \bar{Z}$ is a multifunction with $D(f) = \{x \in X: f(x) \neq \emptyset\}$, $F: D(F) \subseteq X \times Y \rightarrow \bar{Z}$ is a point-to-set map defined on $D(F) = \{(x,y) \in X \times Y: F(x,y) \neq \emptyset\}$ such that $F(x,0) = f(x)$ for every $x \in D(f)$ and we consider the Z -duality $(\cdot, \cdot)$ between $X \times Y$ and $L(X,Z) \times L(Y,Z)$ defined by

$$(1) \quad ((x,y), (T,S)) = Tx + Sy, \quad \forall x \in X, \quad \forall y \in Y, \quad \forall T \in L(X,Z), \quad \forall S \in L(Y,Z)$$

where by $L(X,Z)$ ($L(Y,Z)$) we denote the real linear space of all linear operators $T: X \rightarrow Z$ ($S: Y \rightarrow Z$) with the remark that we shall consider $L(X,Z)$ ($L(Y,Z)$) containing only the linear and continuous operators from X to Z (from Y to Z) whenever X and Z (Y and Z) are linear topological spaces, then we say that the multifunction f is *vectorial subdifferentiable* at a point $x_0 \in D(f)$ if

$$(2) \quad \partial_v f(x_0) = \{T \in L(X,Z): \text{there exists } y_0 \in f(x_0) \text{ such that } Tx_0 - y_0 \neq Tu - y \text{ for every } (u,y) \in G(f)\} \neq \emptyset$$

where $G(f) = \{(x,y): y \in f(x)\}$ is the graph of f and we call the point-to-set map $f^*: D(f^*) \subseteq L(X,Z) \rightarrow \bar{Z}$ defined by

$$(3) \quad f^*(T) = \text{SUP}_1\{Tu - y: (u,y) \in G(f)\}$$

where $D(f^*) = \{T \in L(X,Z): \text{SUP}_1\{Tu - y: (u,y) \in G(f)\} \neq \emptyset\}$ the *vectorial conjugate* of f . It is easy to see that the multifunction f is vectorial subdifferentiable

at $x_0 \in D(f)$ if and only if there exist $y_0 \in f(x_0)$ and $T \in L(X, Z)$ such that $Tx_0 - y_0 \in f^*(T)$.

In accordance with [4], the vectorial conjugate of F with respect to the above Z -duality is the multifunction $F^*: D(F^*) \subseteq L(X, Z) \times L(Y, Z) \rightarrow \bar{Z}$ defined by

$$(4) \quad F^*(T, S) = \sup_1 \{Tx + Sy - z: ((x, y), z) \in G(F)\}$$

where $D(F^*) = \{(T, S) \in L(X, Z) \times L(Y, Z): \sup_1 \{Tx + Sy - z: ((x, y), z) \in G(F)\} \neq \emptyset\}$.

Therefore

$$(5) \quad F^*(0, y^*) = \sup_1 \{y^*y - z: ((x, y), z) \in G(F)\}$$

for every $y^* \in L(Y, Z)$.

We consider the following vectorial problems:

$$(P) \quad \text{MIN}[f(X)] = \text{MIN}(f)$$

and

$$(P^*) \quad \text{MAX} \left[\bigcup_{y^* \in L(Y, Z)} (-F^*(0, y^*)) \right] = \text{MAX}[-F^*(0, \cdot)]$$

and we define

$$H(y) = \inf \left[\bigcup_{(x, y) \in D(F)} F(x, y) \right],$$

$$H_1(y) = \inf_1 \left[\bigcup_{(x, y) \in D(F)} F(x, y) \right],$$

$$\bar{H}_1(y) = \min \left[\bigcup_{(x, y) \in D(F)} F(x, y) \right].$$

Assume that Z is a linear topological space. We say that a nonempty set $A \subseteq Z$ is Z_+ -bounded if there exists a bounded set $A_0 \subset Z$ such that $A \subseteq A_0 + Z_+$, and A is said to be Z_+ -closed if $A + Z_+$ is closed. Moreover, the set A is called Z_+ -compact if it is Z_+ -bounded and Z_+ -closed. If, in addition, Z is a quasi-complete locally convex space ordered by a closed and nuclear cone Z_+ , then the following lemma established in [4] holds.

LEMMA. If $F_1: D(F_1) \subseteq X \rightarrow P(\overline{Z})$, $F_2: D(F_2) \subseteq X \rightarrow P(\overline{Z})$ are multifunctions such that the set $F_2(x)$ is Z_+ -compact for every $x \in D(F_1) \cap D(F_2)$, then

$$(6) \quad \inf_1 \bigcup_{x \in D(F_1) \cap D(F_2)} [F_1(x) + F_2(x)] = \inf_1 \bigcup_{x \in D(F_1) \cap D(F_2)} [F_1(x) + \min F_2(x)].$$

In all our further considerations we suppose that Z is a quasi-complete locally convex space ordered by a closed and nuclear cone Z_+ .

THEOREM 1.

- (i) $\bigcap_{u \in D(H)} [y^*u - H(u)] \cap H_1^*(y^*) \subseteq F^*(0, y^*) \subseteq \sup\{y^*y - z: ((x, y), z) \in G(F)\}$
 $\subseteq \bigcap_{u \in D(H)} [y^*u - H(u)] \cap \sup\{y^*u - H_1(u): u \in D(H_1)\}, \quad \forall y^* \in L(Y, Z);$
- (ii) if the multifunction $F_1: D(F_1) \subseteq Y \rightarrow \overline{Z}$ defined by $F_1(y) = \bigcup_{(x, y) \in D(F)} F(x, y)$ has Z_+ -compact values, then $F^*(0, y^*) = \overline{H}_1^*(y^*)$ for every $(0, y^*) \in D(F^*)$;

(iii) if the assumption of (ii) is satisfied, then

$$(7) \quad \begin{aligned} \max[-F^*(0, .)] &= \max \left[\bigcup_{y^* \in D(\overline{H}_1^*)} (-\overline{H}_1^*(y^*)) \right] \\ &= \overline{H}_1^{**}(0) \cap \left[\bigcup_{y^* \in D(\overline{H}_1^*)} (-\overline{H}_1^*(y^*)) \right]. \end{aligned}$$

PROOF. (i) It is an immediate consequence of the above definitions and discussions. We remark that for this point the hypothesis which precedes the theorem is not necessary.

(ii) By virtue of the above lemma we have

$$\begin{aligned} F^*(0, y^*) &= \sup_1 \{y^*y - z: ((x, y), z) \in G(F)\} \\ &= \sup_1 \left[\bigcup_{\substack{y \in Y \\ (x, y) \in D(F)}} [y^*y - \min F_1(y)] \right] \\ &= \sup_1 \{y^*y - z: (y, z) \in G(\overline{H}_1)\} \\ &= \overline{H}_1^*(y^*), \quad \forall (0, y^*) \in D(F^*). \end{aligned}$$

(iii) It follows from (ii) taking into account the fact that the vectorial biconjugate of $\bar{H}_1$, denoted by $\bar{H}_1^{**}$, is defined by

$$(8) \quad \bar{H}_1^{**}(x) = \sup_1 \{Tx - y^*: (T, y^*) \in G(\bar{H}_1^*)\}$$

for every $x \in D(\bar{H}_1^{**}) = \{x \in X: \sup_1 \{Tx - y^*: (T, y^*) \in G(\bar{H}_1^*)\} \neq \emptyset\}$. This completes the proof.

COROLLARY. If X and Y are linear topological spaces, $D(F)$ is a compact set and F is an upper semicontinuous multifunction with compact values, then $F^*(0, y^*) = \bar{H}_1^*(y^*)$, $\forall (0, y^*) \in D(F^*)$.

PROOF. Recall that F is said to be upper semicontinuous if for each $(x, y) \in D(F)$ and each open set G of Z containing $F(x, y)$, there exists an open set $U \subseteq X \times Y$ containing (x, y) such that $F(U \cap D(F)) \subseteq G$. It is sufficient to prove that the multifunction F_1 defined in (ii) of the above theorem has Z_+ -compact values. Indeed, for every $y \in Y$ for which there exists $x \in X$ with $(x, y) \in D(F)$ the set $K_y = \{(x, y): (x, y) \in D(F)\}$ is compact and $F_1(y) = \bigcup_{(x, y) \in D(F)} F(x, y) = F(K_y)$. Let $\{D_\alpha: \alpha \in A\}$ be an open cover of $F(K_y)$ and $(x, y) \in K_y$. Since $F(x, y)$ is a compact subset of Z and $F(x, y) \subseteq F(K_y)$, there exists a finite subcover $\{D_{\alpha_1}(x, y), D_{\alpha_2}(x, y), \dots, D_{\alpha_n}(x, y)\}$ for $F(x, y)$. If we put $D(x, y) = \bigcup_{i=1}^n D_{\alpha_i}(x, y)$, then $D(x, y)$ is an open set which contains $F(x, y)$ and because F is upper semicontinuous, there exists an open set $G(x, y) \subseteq X \times Y$ containing (x, y) such that $F(G(x, y) \cap D(F)) \subseteq D(x, y)$. The family $\{G(x, y): (x, y) \in K_y\}$ is an open cover of the compact set K_y , hence there exist $(x_1, y), (x_2, y), \dots, (x_m, y)$ in K_y such that $K_y \subseteq \bigcup_{j=1}^m G(x_j, y)$. Therefore

$$F(K_y) \subseteq F\left[\left(\bigcup_{j=1}^m G(x_j, y)\right) \cap D(F)\right] = \bigcup_{j=1}^m F(G(x_j, y) \cap D(F)) \subseteq \bigcup_{j=1}^m D(x_j, y),$$

that is, there exists a finite subcover $\{D_{\alpha_i}(x_j, y): i = \bar{1}, n, j = \bar{1}, m\}$ of $\{D_\alpha: \alpha \in A\}$ for $F(K_y)$. Hence $F_1(y)$ is compact for every $y \in D(F_1)$. Then, F_1 has Z_+ -compact values and the proof is completed.

Our main result is the following.

THEOREM 2.

(i) In the conditions of the point (ii) from Theorem 1 if the multifunction $\bar{H}_1$ is vectorial subdifferentiable at 0, then there exists $\hat{x}$ solution of problem (P) and $\hat{y}^*$ solution of problem (P*) such that $f(\hat{x}) \cap (-F^*(0, \hat{y}^*)) \neq \emptyset$;

(ii) for every $\hat{x} \in D(f)$ and $(0, y^*) \in D(F^*)$ which satisfy the above relation, $\hat{x}$ is solution for (P) and $\hat{y}^*$ is solution for (P*).

PROOF. (i) Since $\bar{H}_1$ is vectorial subdifferentiable at 0, it follows that there exist $y_0^* \in L(Y, Z)$ and $y_0 \in \bar{H}_1(0)$ such that $y_0 \in -\bar{H}_1^*(y_0^*)$. Assume that $y_0 \notin \sup_1 \left[\bigcup_{y^* \in D(\bar{H}_1^*)} (-\bar{H}_1^*(y^*)) \right]$. Then, there exist $y^* \in D(\bar{H}_1^*)$ and $h^* \in \bar{H}_1^*(y^*)$ such that $y_0 < -h^*$, i.e., $h^* < y^*(0) - y_0$, contradiction. Hence

$$y_0 \in \bar{H}_1(0) \cap \left[\bigcup_{y^* \in D(\bar{H}_1^*)} (-\bar{H}_1^*(y^*)) \right] \cap \sup_1 \left[\bigcup_{y^* \in D(\bar{H}_1^*)} (-\bar{H}_1^*(y^*)) \right] = \text{MIN}(f) \cap \text{MAX}[-F^*(0, 1)]$$

and (i) is proved.

(ii) This result is obvious from the definition of the multifunction F^* .

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*Department of Mathematics
5600-Piatra Neamt
R.S. România*

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