

A SYNTHESIS OF BOCHNER AND RIESZ TYPE REPRESENTATION THEOREMS¹

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Sommaire

Soit i) V un treillis vectoriel de fonctions réelles bornées sur un ensemble G , fermé sous les produits et contenant la fonction constante 1 et ii) A l'algèbre d'ensembles dans G engendrée par la classe Z des ensembles zéros (ou bien par la classe Z^c des ensembles co-zéros) des fonctions dans V . Alors, étant donné une fonctionnelle linéaire positive L sur V , il existe une fonction d'ensembles additive $\mu: A \rightarrow [0, \infty)$ qui est régulière de l'intérieur et de l'extérieur par rapport respectivement à Z et Z^c et qui satisfait la relation $Lf = \int_G f d\mu$ ($f \in V$). Ce résultat permet d'établir un lien entre un théorème de Bochner et des théorèmes de représentations plus récents. Il permet également de généraliser en partie une caractérisation des homomorphismes continus due à Sultan ainsi qu'un théorème de représentation démontré par Pollard et Topsøe pour les fonctionnelles linéaires positives ayant la propriété de Daniell, et de fournir une application aux mesures complexes, régulière et boréliennes définies sur les groupes abéliens localement compacts.

1. Introduction and basic results

The following result was obtained by Bochner in 1939 (see [4, pp. 773-775]).

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THEOREM 1 (Bochner). Let V be a real vector lattice, consisting of bounded (real-valued) functions on a set G , closed under products and containing the constant function 1. If L is a positive linear functional on V and J_L is the class of (Jordan) sets E in G (with indicator function I_E) for which

$$\inf\{L(f-g): g \leq I_E \leq f; f, g \in V\} = 0$$

then

- 1) J_L is an algebra of sets,
- 2) for any given $f \in V$, $\{z: f(z) \geq r\}$ lies in J_L for a dense set of reals r , and
- 3) the set function m_L defined on J_L by

$$m_L(E) = \inf\{Lf: I_E \leq f; f \in V\}$$

is a positive finitely additive set function for which

$$Lf = \int_G f \, dm_L$$

for all $f \in V$.

Here, the integral is taken in the sense of Riemann. It is unfortunate that both the algebra J_L of Jordan sets and the dense sets of reals depend on the linear functional L . Our first result, which we now state, eliminates these weaknesses. For this theorem and the remainder of the paper, we refer the reader to [6, pp. 101-125] for the theory of integration with respect to finitely additive set functions. Some of our results were communicated without proof in [3].

THEOREM 2. Let i) V be a real vector lattice consisting of bounded (real-valued) functions on a set G , closed under products and containing the constant function 1, ii) Z and Z^c be the class of zero and cozero sets respectively for all functions in V , and iii) A be the algebra of sets (in G) generated by Z . Then, given a positive linear functional L on V , there exists a positive finitely additive set function μ on A such that

- 1) for all $f \in V$, the following integral exists (since f is the uniform

limit of A -measurable step functions) and is given by

$$Lf = \int_G f \, d\mu$$

and

2) μ is regular in the sense that, for all $E \in A$,

$$\begin{aligned} \mu(E) &= \inf\{\mu(V) : E \text{ in } V, V \in \mathcal{Z}^c\} \\ &= \sup\{\mu(Z) : Z \text{ in } E, Z \in \mathcal{Z}\}. \end{aligned}$$

REMARK 1. a) Let $\mathcal{Q}$ be the class of all quotients $f/(f+g)$ with $f, g \in \mathcal{V}$ positive and $f+g > 0$, and let $\mathcal{V}^*$ be the smallest real linear space, with uniform norm, containing all products hq with $h \in \mathcal{V}$ and $q \in \mathcal{Q}$. Clearly $\mathcal{V}^*$ is closed under products and the zero set of any given function in $\mathcal{V}^*$ lies in $\mathcal{Z}$. Note that if $\mathcal{V}$ is the set of all continuous real-valued almost periodic functions on the real line, then it is a vector lattice properly contained in $\mathcal{V}^*$. In fact, the function $(\cos^2 t)/(\cos^2 t + \cos^2 \pi t)$ which lies in $\mathcal{V}^*$, is not uniformly continuous and so not almost periodic.

b) Given any two disjoint sets $Z_1, Z_2 \in \mathcal{Z}$ and any two functions $h_1, h_2 \in \mathcal{V}$ such that $Z_i = h_i^{-1}(\{0\})$ ($i = 1, 2$), then the function f given by $h_1^2/(h_1^2 + h_2^2)$ lies in $\mathcal{Q}$ (and so in $\mathcal{V}^*$) and has the property that $f = 0$ on Z_1 and $f = 1$ on Z_2 (i.e., $\mathcal{V}^*$ separates disjoint sets in $\mathcal{Z}$).

c) The use of a positive extension of L from $\mathcal{V}$ to $\mathcal{V}^*$ in the proof of Theorem 2 removes any possibility for uniqueness in general. In view of the following corollary and the fact that $\mathcal{V} = \mathcal{V}^*$ when $\mathcal{V}$ consists of all bounded continuous real-valued functions on a topological space G , we get that our result contains the well-known representation theorem of Alexandroff [1]. It also generalizes part of a theorem due to Kirk [8, p. 461] (see also [12, pp. 62, 63]) since it is not assumed that i) $\mathcal{V}$ separates points of G , and ii) that $\mathcal{V}$ separates disjoint sets in $\mathcal{Z}$. Theorem 2 can be proved from the very general results of Pollard and Topsøe (see [10] and [13]) by showing that the vector lattice generated by $\mathcal{V}^*$ has the same class of zero sets $\mathcal{Z}$ and that disjoint zero sets are separated by the functions

in it. Since the proof of their result rests on previous work by Topsøe, we provide here a self contained proof of a more general theorem (see Theorem 3) which establishes a relationship between Theorem 2 and Theorem 1 of Bochner.

d) For the case when G is a locally compact abelian group, Theorem 2 is applied in [3, p. 153] to obtain integral representations for i) positive linear functionals on the space of almost periodic functions and ii) not necessarily continuous positive definite functions.

COROLLARY 1. Given a positive linear functional L^* on V^* , there exists a unique positive finitely additive set function μ on A such that

1) for all $f \in V^*$, the following integral exists (since f is the uniform limit of A -measurable step functions) and is given by

$$L^*f = \int_G f d\mu$$

and

2) μ is regular in the sense that, for all $E \in A$,

$$\begin{aligned} \mu(E) &= \inf\{\mu(V) : E \text{ in } V, V \in Z^c\} \\ &= \sup\{\mu(Z) : Z \text{ in } E, Z \in Z\}. \end{aligned}$$

Under the hypothesis of Theorem 2, there exists a (not necessarily unique) positive linear functional L^* which extends L from V to V^* (see [5, p. 144]). Let μ be the positive function defined on the class P of all subsets of G by

$$\mu(V) = \sup\{L^*f : 0 \leq f \leq I_V; f \in V^*\}$$

for all $V \in Z^c$, and

$$\mu(E) = \inf\{\mu(V) : E \text{ in } V; V \in Z^c\}$$

for all $E \in P$. Clearly,

$$\mu(E_1) \leq \mu(E_2)$$

whenever E_1 is contained in E_2 . Let A_{L^*} denote the class of sets $E \in P$ for which

$$\mu(E) = \sup\{\mu(Z) : Z \text{ in } E; Z \in \mathcal{Z}\}.$$

Clearly $\mathcal{Z}$ lies in A_{L^*} .

Theorem 2 is contained in the following useful generalization of Theorem 1 that we prove in the next section.

THEOREM 3. A_{L^*} is an algebra of sets in G containing both J_L and A . Furthermore, $\mu = m_L$ on J_L , all functions in V^* are uniform limits of A_{L^*} -measurable step functions and μ is the unique positive finitely additive set function on A_{L^*} such that

1) for all $f \in V^*$, we have

$$L^*f = \int_G f \, d\mu$$

and

2) μ is regular in the sense that, for all

$$\begin{aligned} \mu(E) &= \inf\{\mu(V) : E \text{ in } V; V \in \mathcal{Z}^c\} \\ &= \sup\{\mu(Z) : Z \text{ in } E; Z \in \mathcal{Z}\}. \end{aligned}$$

We also have, for all $Z \in \mathcal{Z}$,

$$\mu(Z) = \inf\{L^*q : I_Z \leq q; q \in \mathcal{Q}\}$$

and, for all $V \in \mathcal{Z}^c$,

$$\mu(V) = \sup\{L^*q : q \leq I_V; q \in \mathcal{Q}\}.$$

REMARK 2. Let G be the real line, V be the vector lattice consisting of all continuous real-valued almost periodic functions on G , and L be the positive linear functional on V given by

$$Lf = \lim_{T \rightarrow \infty} \int_{-T}^T f(t) \, dt.$$

Note that L does not have the Daniell property and that all bounded subsets of G lie in J_L . So this is an example where A (which is a subalgebra of the Borel sets

in G is properly contained in A_{L*} . Hence, Theorem 3 cannot be deduced from the results of Pollard and Topsøe (see [10] and [13]). Nor can it be shown by Theorem 1 of Bochner since in some cases there are sets in Z which are not in J_L (see [4, p. 777]).

We also have the following corollary which contains part of a result first obtained by Pollard and Topsøe [10, p. 181] (see also [13, p. 243]).

COROLLARY 2. If V be a Banach algebra (with uniform norm), then given a positive linear functional L on V with the Daniell property (i.e., $Lf_n \rightarrow 0$ for all sequences $f_n \rightarrow 0$ in V), there exists a unique positive linear extension L^* of L to V^* with the same property. Furthermore, for this extension, A_{L*} is a σ -algebra on which is defined a unique positive countably additive set function μ such that

1) for all $f \in V$, the following integral exists in the sense of Lebesgue and is given by

$$Lf = \int_G f \, d\mu$$

and

2) μ is regular in the sense that, for all $E \in A_{L*}$,

$$\begin{aligned} \mu(E) &= \inf\{\mu(V): E \text{ in } V, V \in Z^c\} \\ &= \sup\{\mu(Z): Z \text{ in } E, Z \in Z\}. \end{aligned}$$

We also have, for all $V \in Z^c$,

$$\mu(V) = \sup\{Lf: 0 \leq f \leq I_V; f \in V\}$$

and for all $Z \in Z$,

$$\mu(Z) = \inf\{Lf: I_Z \leq f \leq 1; f \in V\}.$$

REMARK 3. Note that A_{L*} is not in general the same as the σ -algebra S generated by Z . For example, if V is the class of all continuous real-valued functions on the interval $[0,1]$ and L be the positive linear functional on V defined so that Lf is the Lebesgue integral of f on $[0,1]$, then A_{L*} consists

of all Lebesgue sets in $[0,1]$ (see for example [11, pp. 52-54]) while S consists of all Borel sets. Hence S is properly contained in A_{L^*} and so Corollary 2 cannot be proved by means of the results of Pollard and Topsøe (see [10, p. 181] and [13, p. 243]). Furthermore our corollary cannot be deduced from the Daniell extension theorem (see for example [9, p. 35]) since, in the latter, the set function representing the linear functional is not necessarily unique and has regularity properties with respect to a class of sets generally much larger than the zero and co-zero sets of functions in V .

Let H^* denote the set of all continuous (i.e., positive) real-valued homomorphisms defined on V^* . Without assuming that Z is closed under countable intersections, as is done in [12, pp. 59-61], we obtain the following characterization of H^* .

COROLLARY 3. *For a linear functional L^* to be in H^* it is necessary and sufficient that there exist a unique finitely additive set function $\mu: A \rightarrow \{0,1\}$ which is inner and outer regular with respect to sets in Z and Z^c respectively, and is such that*

$$L^*f = \int_G f \, d\mu$$

for all $f \in V^*$.

REMARK 4. As in [12], let us write $IR(G)$ for the class of all set functions defined on A as in Corollary 3. If we give $IR(G)$ the vague topology (i.e., a sequence $\{\mu_i\}$ in $IR(G)$ converges to $\mu \in IR(G)$ if and only if $\int f \, d\mu_i \rightarrow \int f \, d\mu$ for all $f \in V^*$) and H^* the weak star topology, then the bijection defined by the correspondence in Corollary 3 is a homeomorphism.

2. Proof of our basic results

As will be evident from the proof, Theorem 2 is contained in Theorem 3 in the sense that the set function in the first is the restriction on A of that in the second. We now break up the proof of our results into a sequence of steps

typical of such representation theorems (see, for example, the proof of Alexandroff's representation theorem in [14, pp. 115-117]).

STEP I. Given $Z \in \mathcal{Z}$ and $V \in \mathcal{Z}^c$, and given $f, g \in \mathcal{V}^*$ with $g \geq I_Z$ and $f \leq I_V$, we have $L^*g \geq \mu(Z)$ and $L^*f \leq \mu(V)$.

PROOF. Given $\varepsilon > 0$, put

$$V_\varepsilon = \{x \in G: (1+\varepsilon)g(x) > 1\}.$$

Clearly Z is contained in V_ε . Furthermore, for all $h \in \mathcal{V}^*$ with $0 \leq h \leq I_{V_\varepsilon}$ we have $h \leq (1+\varepsilon)g$ (and so $L^*h \leq (1+\varepsilon)L^*g$). Hence

$$\mu(Z) \leq \mu(V_\varepsilon) \leq (1+\varepsilon)L^*g$$

and so $\mu(Z) \leq L^*g$.

We now prove the second inequality. Note that $1-f \geq I_{G \setminus V}$. So, by above, we have that $L^*(1-f) \geq \mu(G \setminus V)$. Now, choose $V' \in \mathcal{Z}^c$ containing $G \setminus V$, such that $\mu(V') < \mu(G \setminus V) + \varepsilon$. Then $L^*(1-f) > \mu(V') - \varepsilon$. Hence, $L^*f < L^*1 - \mu(V') + \varepsilon$. Thus, for all $h \in \mathcal{V}^*$ such that $I_{G \setminus V} \leq h \leq I_V$, we have $L^*f < L^*(1-h) + \varepsilon$. Now, $0 \leq 1-h \leq I_V$. Hence, it follows that $L^*f < \mu(V) + \varepsilon$. This completes the proof.

STEP II. Given $Z \in \mathcal{Z}$ contained in $V \in \mathcal{Z}^c$, we have

$$\mu(V \setminus Z) \leq \mu(V) - \mu(Z).$$

PROOF. Note that $V \setminus Z \in \mathcal{Z}^c$. Hence, given $\varepsilon > 0$, there exists $h \in \mathcal{V}^*$ such that $0 \leq h \leq I_{V \setminus Z}$ and

$$L^*h > \mu(V \setminus Z) - \varepsilon.$$

Let $V' \in \mathcal{Z}^c$ denote the intersection of V with $h^{-1}([0, \varepsilon))$. Then V' contains Z . By Remark 1b), there exists $f \in \mathcal{V}^*$ such that $I_Z \leq f \leq I_{V'}$. Note that $f+h-\varepsilon < I_V$. So, by Step I and the above, we have

$$\begin{aligned}
\mu(V) &\geq L^*(f+h-\varepsilon) \\
&= L^*f + L^*h - \varepsilon L^*1 \\
&> \mu(Z) + \mu(V \setminus Z) - \varepsilon(1+L^*1).
\end{aligned}$$

Hence,

$$\mu(V) \geq \mu(Z) + \mu(V \setminus Z).$$

The proof is complete.

STEP III. Z^C lies in A_{L^*} .

PROOF. Given $V \in Z^C$ and $\varepsilon > 0$, choose $f \in \mathcal{V}^*$ such that $0 \leq f \leq I_V$ and $L^*f > \mu(V) - \varepsilon$. Put $Z = f^{-1}([\varepsilon, 1])$. Then Z is contained in V and $f - \varepsilon < I_Z$. So, by Step I we get

$$L^*f - \varepsilon L^*1 \leq \inf\{\mu(V') : Z \text{ in } V', V' \in Z^C\} = \mu(Z).$$

Hence,

$$\mu(Z) \leq \mu(V) < L^*f + \varepsilon \leq \mu(Z) + \varepsilon(1+L^*1).$$

This completes the proof.

STEP IV. $\mu(\bigcup E_i) \leq \sum \mu(E_i)$ for all finite sequences $E_1, \dots, E_n \in \mathcal{P}$.

PROOF. Given $\varepsilon > 0$, there exist sets $V_1, \dots, V_n \in Z^C$ with E_i in V_i and $\mu(V_i) < \mu(E_i) + \varepsilon/n$ ($i = 1, \dots, n$). Let 1) $h_i \in \mathcal{V}$ be such that $0 \leq h_i \leq 1$ and $V_i = h_i^{-1}((0, 1])$, 2) $V = \bigcup V_i$, and 3) $f \in \mathcal{V}^*$ be such that $0 \leq f \leq I_V$. There also exists a set $Z \in Z$ with Z in V and $\mu(V) < \mu(Z) + \varepsilon$. So, by Step II we have $\mu(V \setminus Z) < \varepsilon$. Let $h \in \mathcal{V}$ be such that $0 \leq h \leq 1$ and $G \setminus Z = h^{-1}((0, 1])$. Then the functions $f_i: G \rightarrow [0, 1]$ ($i = 1, \dots, n$) defined by

$$f_i = fh_i / (\sum h_j + h)$$

and the function $f': G \rightarrow [0, 1]$ defined by

$$f' = fh / (\sum h_j + h)$$

lie in $\mathcal{V}^*$. Now $f = \sum f_i + f'$ and so

$$\begin{aligned} L^*f &= \sum L^*f_i + L^*f' \\ &\leq \sum \mu(V_i) + \mu(V \setminus Z) \\ &< \sum \mu(E_i) + 2\epsilon. \end{aligned}$$

This completes the proof.

STEP V. $\mu(Z_1 \cup Z_2) = \mu(Z_1) + \mu(Z_2)$ for all disjoint $Z_1, Z_2 \in \mathcal{Z}$.

PROOF. By Remark 1b) there exists $f: G \rightarrow [0,1]$ in $\mathcal{V}^*$ such that $f = 1$ on Z_1 and $f = 0$ on Z_2 . Let $V_1 = f^{-1}((\frac{1}{2}, 1])$ and $V_2 = f^{-1}([0, \frac{1}{2}))$. Then V_1, V_2 are disjoint elements of $\mathcal{Z}^c$ with Z_i in V_i ($i = 1, 2$).

Given $\epsilon > 0$, choose $V \in \mathcal{Z}^c$ with $Z_1 \cup Z_2$ in V and $\mu(V) < \mu(Z_1 \cup Z_2) + \epsilon/3$. Also, choose $f_1, f_2 \in \mathcal{V}^*$ with $0 \leq f_i \leq I_{V_i \cap V}$ and $L^*f_i > \mu(V_i \cap V) - \epsilon/3$ ($i = 1, 2$). Then $0 \leq f_1 + f_2 \leq I_V$ and $\mu(Z_1) + \mu(Z_2) \leq \mu(V_1 \cap V) + \mu(V_2 \cap V) < L^*f_1 + L^*f_2 + 2\epsilon/3 = L^*(f_1 + f_2) + 2\epsilon/3 \leq \mu(V) + 2\epsilon/3 < \mu(Z_1 \cup Z_2) + \epsilon$. So, by Step IV, we get $\mu(Z_1 \cup Z_2) = \mu(Z_1) + \mu(Z_2)$. This completes the proof.

STEP VI. A_{L^*} is an algebra of sets containing $\mathcal{A}$ on which μ is additive.

PROOF. a) A_{L^*} is closed under disjoint unions and μ is additive. To see this, let E_1, E_2 be two disjoint sets in A_{L^*} and choose $\epsilon > 0$. There exist Z_i in E_i such that $\mu(E_i) < \mu(Z_i) + \frac{1}{2}\epsilon$ ($i = 1, 2$). Put $Z = Z_1 \cup Z_2$. Then $\mu(E_1 \cup E_2) \geq \mu(Z) = \mu(Z_1) + \mu(Z_2) > \mu(E_1) + \mu(E_2) - \epsilon$ and so, by Step IV, we have $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. This shows that $\mu(Z) > \mu(E_1 \cup E_2) - \epsilon$. Since Z lies in $E_1 \cup E_2$, we have that A_{L^*} is closed under disjoint unions and so μ is additive on A_{L^*} .

b) If $E_1, E_2 \in A_{L^*}$, then $E_1 \setminus E_2 \in A_{L^*}$ as we now show. Given $\epsilon > 0$, choose $Z_i \in \mathcal{Z}$ and $V_i \in \mathcal{Z}^c$ such that Z_i lies in E_i , E_i lies in V_i and $\mu(E_i) - \frac{1}{2}\epsilon < \mu(Z_i) \leq \mu(V_i) < \mu(E_i) + \frac{1}{2}\epsilon$ ($i = 1, 2$). Then $E_1 \setminus E_2$ lies in $(V_1 \setminus Z_1) \cup (Z_1 \setminus V_2) \cup (V_2 \setminus Z_2)$, and since $\mu(V_i \setminus Z_i) = \mu(V_i) - \mu(Z_i) < \epsilon$, we have $\mu(E_1 \setminus E_2) < \mu(Z_1 \setminus V_2) + 2\epsilon$ where $Z_1 \setminus V_2 \in \mathcal{Z}$ and $Z_1 \setminus V_2$ lies in $E_1 \setminus E_2$.

It now follows from the fact that A_{L^*} contains Z and by parts a) and b) that A_{L^*} is an algebra of sets containing A on which μ is additive. The proof is complete.

STEP VII. For all $f \in \mathcal{V}^*$ we have

$$L^*f = \int_G f \, d\mu.$$

PROOF. Assume for the moment that $f > 0$ and take $b > 0$ such that $G = f^{-1}((0, b))$. Given $\varepsilon > 0$, let $0 = y_0 < y_1 < \dots < y_n = b$ be a partition of $[0, b]$ such that $y_i - y_{i-1} < \varepsilon$ ($i = 1, \dots, n$). Clearly, the sets $E_i = f^{-1}([y_{i-1}, y_i])$ ($i = 1, \dots, n$) lie in A . Let $V_i \in \mathcal{Z}^C$ ($i = 1, \dots, n$) be such that E_i lies in V_i and $\mu(V_i) < \mu(E_i) + \varepsilon/n$ and $f < y_i + \varepsilon$ on V_i . Now choose nonnegative functions $g_i \in \mathcal{V}$ ($i = 1, \dots, n$) such that $V_i = g_i^{-1}((0, \infty))$ and let h_i be those functions in $\mathcal{V}^*$ defined by

$$h_i = g_i / \sum g_j.$$

Clearly $0 \leq h_i \leq 1_{V_i}$ and $f = \sum (fh_i)$. Furthermore, $fh_i \leq (y_i + \varepsilon)h_i$ and, on E_i , we have $f \geq y_{i-1} > y_i - \varepsilon$. Hence, we have

$$\begin{aligned} L^*f &= \sum L^*(fh_i) \\ &\leq \sum (y_i + \varepsilon)L^*h_i \\ &\leq \sum (y_i + \varepsilon)\mu(V_i) \\ &< \sum (y_i + \varepsilon)\mu(E_i) + \sum (y_i + \varepsilon)\varepsilon/n \\ &= \sum (y_i - \varepsilon)\mu(E_i) + 2\varepsilon \sum \mu(E_i) + \varepsilon(\sum y_i/n + \varepsilon) \\ &< \sum y_{i-1}\mu(E_i) + 2\varepsilon\mu(G) + \varepsilon(b + \varepsilon) \\ &\leq \int_G f \, d\mu + \varepsilon(2\mu(G) + b + \varepsilon) \end{aligned}$$

and so

$$L^*f \leq \int_G f \, d\mu.$$

Now, any real-valued function $f \in \mathcal{V}^*$ can be made strictly positive by adding a constant and so, since L^* is linear, the above inequality also holds for all real-valued $f \in \mathcal{V}^*$. Substituting $-f$ for f we get

$$\int_G f \, d\mu \leq L^*f$$

and so

$$\int_G f \, d\mu = L^*f$$

for all $f \in V^*$. This completes the proof.

It now follows that

$$Lf = \int_G f \, d\mu$$

for all $f \in V$.

STEP VIII. μ is unique on A_{L^*} .

PROOF. Suppose μ' also satisfies Theorem 3. By regularity, it is sufficient to show that $\mu(Z) = \mu'(Z)$ for all $Z \in \mathcal{Z}$. Fix $Z \in \mathcal{Z}$ and choose $\epsilon > 0$. There exists $V \in \mathcal{Z}^c$ such that Z lies in V and $\mu'(V) < \mu'(Z) + \epsilon$. By Remark 1b), there exists $f \in V^*$ such that $I_Z \leq f \leq I_V$. So we have $\mu(Z) \leq L^*f \leq \mu'(V) < \mu'(Z) + \epsilon$. Hence $\mu(Z) \leq \mu'(Z)$. Interchanging μ with μ' we also have $\mu'(Z) \leq \mu(Z)$. So it follows that $\mu(Z) = \mu'(Z)$. This completes the proof.

STEP IX. J_L is a subalgebra of A_{L^*} on which $\mu = m_L$.

PROOF. Suppose that $E \in J_L$. Then

$$\inf\{L(f-g): g \leq I_E \leq f; f, g \in V\} = 0.$$

So, given $\epsilon > 0$, there exists $f \in V$ such that $I_E \leq f$ and $Lf < \mu(E) + \epsilon$. Hence $\mu(E) = m_L(E)$ since

$$\begin{aligned} \mu(E) &= \inf\{Lf: I_E \leq f; f \in V\} \\ &= m_L(E). \end{aligned}$$

Similarly, there exists $g \in V$ such that $g \leq I_E$ and $Lg > \mu(E) - \epsilon$. Put $Z = g^{-1}([\epsilon, 1])$. Then $\mu(E) \geq \mu(Z) \geq L(g - \epsilon) = Lg - \epsilon L1 > \mu(E) - \epsilon(1 + L1)$ and so

$$\mu(E) = \sup\{\mu(Z): Z \text{ in } E; Z \in \mathcal{Z}\}.$$

So $E \in A_{L^*}$. Hence, A_{L^*} contains J_L .

STEP X. L^* is uniquely determined by its restriction on Q . Furthermore, for all $z \in Z$ we have

$$\mu(Z) = \inf\{L^*q: I_Z \leq q; q \in Q\}$$

and, for all $V \in Z^C$,

$$\mu(V) = \sup\{L^*q: q \leq I_V; q \in Q\}.$$

PROOF. By Remark 1b), given $Z \in Z$ and $V \in Z^C$ such that Z is contained in V , there exists $q \in Q$ with $I_Z \leq q \leq I_V$. So, it follows that $\mu(Z) \leq L^*q \leq \mu(V)$ and so, by regularity, μ is completely determined on A by the restriction of L^* to Q and consequently L^* is uniquely determined by its restriction on Q . This proves the step.

We have proved Theorem 3 (and so Theorem 2). To show Corollary 1, suppose that μ' is another positive regular finitely additive set function on A for which $L^*f = \int_G f d\mu'$. Then using the ideas in the proof of Step VIII we get our result. We now prove Corollary 2 in three steps.

STEP XI. If L has the Daniell property, then there exists a unique L^* with the same property.

PROOF. Let $\mathcal{V}_1$ be the class of all functions $f: G \rightarrow (-\infty, \infty]$ which are the limit of increasing sequences $\{f_n\}$ in $\mathcal{V}$, and put $L_1f = \lim Lf_n$. Note that L_1 is a well-defined additive extension of L to $\mathcal{V}_1$ with the property that $L_1f \geq L_1g$ for all $f, g \in \mathcal{V}_1$ with $f \geq g$. Let $\mathcal{V}_2$ be the class of all real-valued functions h on G such that

$$\inf\{L_1g: g \geq h; g \in \mathcal{V}_1\} = -\inf\{L_1g: -g \leq h; g \in \mathcal{V}_1\} < \infty.$$

For all $h \in \mathcal{V}_2$, put $L_2h = \inf\{L_1g: g \geq h; g \in \mathcal{V}_1\}$. Since $\mathcal{V}$ is a vector lattice, it follows that $\mathcal{V}_2$ is also a vector lattice containing $\{f \in \mathcal{V}_1: f < \infty; L_1f < \infty\}$ and that L_2 is a positive linear extension of L to $\mathcal{V}_2$ with the Daniell property (see, for example, [9, pp. 29-34]).

Let us show that V^* lies in V_2 . This we do by showing that $hf/(f+g)$ lies in V_2 whenever $f, g, h \in V$ are positive and $1 > f+g > 0$. Clearly, the sequence of functions f_n given by

$$f_n = \sum_{k=0}^n hf(1-f-g)^k$$

lies in V and increases to $hf/(f+g)$. So $hf/(f+g)$ lies in V_1 and $Lh = L_1h = L_1(hf/(f+g)) + L_1(hg/(f+g))$ from which follows that $L_1(hf/(f+g)) < \infty$. Hence, $hf/(f+g)$ lies in V_2 and so V^* is contained in V_2 . So taking for L^* the restriction of L_2 on V^* , we obtain a positive linear functional on V^* with the Daniell property.

Let L^{**} be another extension of L to V^* with the Daniell property. Then, for all positive $f, g, h \in V$ such that $0 < f+g < 1$, we get

$$L^{**}(hf/(f+g)) = \lim_n \sum_{k=0}^n L(hf(1-f-g)^k) = L^*(hf/(f+g))$$

and so $L^{**} = L^*$ on V^* (i.e., there is only one L^* with the Daniell property).

STEP XII. If V is a Banach algebra and L has the Daniell property, then

$$\mu(V) = \sup\{Lf: 0 \leq f \leq I_V; f \in V\}$$

and

$$\mu(Z) = \inf\{Lf: I_Z \leq f \leq 1; f \in V\}$$

for all $V \in Z^c$ and all $Z \in Z$. Also, for all sequences $\{E_i: i = 1, 2, \dots\}$ in $\mathcal{P}$ we have

$$\mu(\bigcup E_i) \leq \sum \mu(E_i).$$

PROOF. The first part of the step follows from

$$\mu(V) = \sup\{L^*q: q \leq I_V; q \in \mathcal{Q}\},$$

$$\mu(Z) = \inf\{L^*q: I_Z \leq q; q \in \mathcal{Q}\}$$

and the fact that, since L has the Daniell property, then L^* can be chosen with the same property and so

$$L^*(f/(f+g)) = \lim_{n \rightarrow \infty} L\left(\sum_{k=0}^n f(1-f-g)^k\right)$$

for all positive $f, g \in \mathcal{V}$ such that $0 < f+g < 1$.

We now prove the countable subadditivity of μ on $\mathcal{P}$. Given $\varepsilon > 0$, let $\{V_i: i = 1, 2, \dots\}$ be a sequence in $\mathcal{Z}^C$ such that, for each i , V_i contains E_i and $\mu(V_i) < \mu(E_i) + \varepsilon/2^{i+1}$. Choose $h_i \in \mathcal{V}$ such that $0 \leq h_i \leq 1/2^{i+1}$ and $V_i = h_i^{-1}((0,1])$. Let $V = \bigcup V_i$ and take $f \in \mathcal{V}$ with $0 \leq f \leq I_V$. Since $\sum h_i \in \mathcal{V}$ and $V = (\sum h_i)^{-1}((0,1])$, we have $V \in \mathcal{Z}^C$. Choose a zero set Z contained in V such that $\mu(V) < \mu(Z) + \varepsilon/2$ (i.e., $\mu(V \setminus Z) < \frac{1}{2}\varepsilon$) and choose $h \in \mathcal{V}$ such that $0 \leq h \leq 1$ and $G \setminus Z = h^{-1}((0,1])$.

Now the functions $f_i: G \rightarrow [0,1]$ given by

$$f_i = fh_i / (\sum h_j + h)$$

and the function $f': G \rightarrow [0,1]$ defined by

$$f' = fh / (\sum h_j + h)$$

are elements of $\mathcal{V}^*$. Furthermore, $f = \sum f_i + f'$ and so, having chosen L^* with the Daniell property, we get

$$Lf = \sum L^*f_i + L^*f' \leq \sum \mu(V_i) + \mu(V \setminus Z) < \sum \mu(E_i) + \varepsilon.$$

So

$$\mu(\bigcup E_i) \leq \mu(\bigcup V_i) = \mu(V) = \sup\{Lf: 0 \leq f \leq I_V, f \in \mathcal{V}\} \leq \sum \mu(E_i).$$

We have proved the step.

STEP XIII. If $\mathcal{V}$ is a Banach algebra and L has the Daniell property then choosing L^* with the same property, we have that A_{L^*} is a σ -algebra on which μ is countably additive and unique.

PROOF. Take $E_1, E_2, E_3, \dots$ a sequence of mutually disjoint sets in A_{L^*} and write E for their union. By Step XII and the fact that $\mu(E) \geq \sum_{i=1}^n \mu(E_i)$ for all

$n = 1, 2, 3, \dots$ we have that $\mu(E) = \sum \mu(E_i)$.

We now show that E lies in A_{L^*} . Given $\varepsilon > 0$, for each $i = 1, 2, 3, \dots$ choose a zero set Z_i in E_i such that $\mu(E_i) < \mu(Z_i) + \varepsilon/2^{i+1}$. Now choose n so that $\sum_{i=1}^n \mu(Z_i) > \sum_{i=1}^{\infty} \mu(Z_i) - \frac{1}{2}\varepsilon$ and let Z denote the union of $Z_1, \dots, Z_n$. Then

$$\mu(Z) > \sum_{i=1}^{\infty} \mu(Z_i) - \frac{1}{2}\varepsilon > \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon = \mu(E) - \varepsilon.$$

Hence, $\mu(E) = \sup\{\mu(Z) : Z \text{ in } E; Z \in \mathcal{Z}\}$ and so $E \in A_{L^*}$.

Uniqueness follows from Step VIII and the fact that L^* is the only positive linear extension of L to $\mathcal{V}^*$ with the Daniell property. This completes the proof.

Finally, we prove Corollary 3 in two steps. Let T be defined on $IR(G)$ by

$$T(\mu)f = \int_G f \, d\mu$$

for all $f \in \mathcal{V}^*$.

STEP XIV. $T: IR(G) \rightarrow H^*$.

PROOF. Clearly if $\mu = 0$, then $T\mu \in H^*$. We show that if $\mu \in IR(G)$ is such that $\mu \neq 0$, then $T\mu$ lies in the subset of H^* consisting of the weak star closure, in the topological dual of $\mathcal{V}^*$, of the class of all positive real-valued homomorphisms L_z^* on $\mathcal{V}^*$ given by $L_z^*f = f(z)$ for all $z \in G$. Choose functions $f_1, \dots, f_n$ in $\mathcal{V}^*$ such that, without loss of generality, $0 \leq f_i < 1$ ($i = 1, \dots, n$). Given an integer $m \geq 0$, put

$$E(i, m, k) = \{z \in G : k \leq 2^m f_i(z) < k+1\}$$

for all $k = 0, \dots, 2^m - 1$, and let the step function $s_{i, m}$ be given by

$$2^m s_{i, m} = \sum_{k=0}^{2^m-1} k I_{E(i, m, k)}.$$

Then

$$\int_G f_i \, d\mu = \lim_{m \rightarrow \infty} \int_G s_{i,m} \, d\mu = \sup_m \sum_{k=0}^{2^m-1} k\mu(E(i,m,k))/2^m.$$

Choose m' large enough so that $(\frac{1}{2})^{m'} < \frac{1}{2}\epsilon$ and

$$\left| \int_G f_i \, d\mu - \sum_{k=0}^{2^{m'}-1} k\mu(E(i,m',k))/2^{m'} \right| < \frac{1}{2}\epsilon$$

for all i . Now, for each i , $\{E(i,m',k): 0 \leq k \leq 2^{m'}-1\}$ is a partition of G . So, taking as partition of G , the common refinement of these n partitions (i.e., the class of all sets $\bigcap_{j=1}^n E(j,m',k_j)$, for all sequences $\{k_j: j = 1, \dots, n\}$ in the set $\{0, \dots, 2^{m'}-1\}$) we get that there exists a finite sequence $\{k_i: i = 1, \dots, n\}$ in $\{0, \dots, 2^{m'}-1\}$ such that $\mu(\bigcap_{i=1}^n E(i,m',k_i)) = 1$. Hence, for each i , we have $\mu(E(i,m',k_i)) = 1$ and $\mu(E(i,m',k)) = 0$ for all $k \neq k_i$. Choose z in $\bigcap_{i=1}^n E(i,m',k_i)$. Then for all $i = 1, \dots, n$ we have

$$\begin{aligned} |L_z^* f_i - \int_G f_i \, d\mu| &\leq |f_i(z) - \sum_{k=0}^{2^{m'}-1} k\mu(E(i,m',k))/2^{m'}| \\ &\quad + \left| \sum_{k=0}^{2^{m'}-1} k\mu(E(i,m',k))/2^{m'} - \int_G f_i \, d\mu \right| \\ &< |f_i(z) - k_i/2^{m'}| + \frac{1}{2}\epsilon \\ &< \epsilon. \end{aligned}$$

STEP XV. T is bijective.

PROOF. Given $L^* \in H^*$, by Remark 1b) and Theorem 3, there exists a unique regular finitely additive set function $\mu: A \rightarrow [0, \infty)$ representing L^* , and this μ is such that for any given set $Z \in \mathcal{Z}$ contained in a set $V \in \mathcal{Z}^c$, there exists $f \in \mathcal{V}^*$ for which $I_Z \leq f \leq I_V$ and $\mu(Z) \leq L^*f \leq \mu(V)$. Since L^* is a homomorphism and since $I_Z \leq f^n \leq I_V$ for all $n = 1, 2, \dots$, we get by regularity that $\mu(Z)$ is either 0 or 1 for all $Z \in \mathcal{Z}$. So, it follows that $\mu \in \text{IR}(G)$.

3. Bounded linear functionals

Let i) $FA(A)$ designate the class of all differences of two positive regular finitely additive set functions on A , and ii) $B(V^*)$ denote the class of all bounded linear functionals on V^* . Given $\mu \in FA(A)$ and $L^* \in B(V^*)$, let $\|\mu\|$ denote the total variation of μ and $\|L^*\|$ the norm of L^* . We now have the following result.

THEOREM 4. Let T be the linear transformation defined on $FA(A)$ by

$$T(\mu)f = \int_G f \, d\mu$$

for all μ in $FA(A)$, and all f in V^* . Then T is a norm preserving bijection of $FA(A)$ onto $B(V^*)$.

PROOF. Clearly T maps $FA(A)$ into $B(V^*)$ and $\|T\mu\| \leq \|\mu\|$. We will prove that T is bijective and that $\|T\mu\| \geq \|\mu\|$ for all $\mu \in FA(A)$.

Let us first show that T is surjective. Given $L^* \in B(V^*)$, let L_+^* and L_-^* be positive linear functionals on V^* for which $L^* = L_+^* - L_-^*$ (see [9, p. 40]). By Corollary 1, there exist positive set functions $\mu^+, \mu^- \in FA(A)$ such that $T\mu^+ = L_+^*$ and $T\mu^- = L_-^*$. So $\mu^+ - \mu^-$ lies in $FA(A)$ and $T(\mu^+ - \mu^-) = L^*$.

Next, let us show that T is injective. Choose $\mu \in FA(A)$ such that $T\mu = 0$ and choose positive set functions $\mu_1^+, \mu_2^- \in FA(A)$ such that $\mu = \mu_1^+ - \mu_2^-$. Then $T\mu_1^+ = T\mu_2^-$ and so, by Corollary 1, we have that $\mu_1^+ = \mu_2^-$ (i.e., $\mu = 0$).

We now show that $\|T\mu\| = \|\mu\|$. Let L_1^* and L_2^* be positive linear functionals on V^* such that $T\mu = L_1^* - L_2^*$ and $\|T\mu\| = L_1^*(1) + L_2^*(1)$ (see for example [9, p. 40]). Since T is surjective, there exist $\mu_1, \mu_2 \in FA(A)$ such that $T\mu_1 = L_1^*$ and $T\mu_2 = L_2^*$. Hence, $\mu = \mu_1 - \mu_2$ since T is injective, and so $\|T\mu\| = T(\mu_1)1 + T(\mu_2)1 = \mu_1(G) + \mu_2(G) \geq \|\mu\|$. This completes the proof.

REMARK 5. Clearly we can prove a similar result when $FA(A)$ is the class of all complex-valued set functions whose real and imaginary parts are each the

difference of two positive regular finitely additive set functions on A , and $B(V^*)$ is the set of all bounded complex-valued linear functionals on V^* .

4. Applications to Borel measures on groups

In this section, G will be a locally compact abelian group (written additively) with identity 0 and dual group $G^\wedge$. Let $\langle z, z^\wedge \rangle$ denote the character $z^\wedge \in G^\wedge$ evaluated at $z \in G$. Given m in the space $M(G)$ consisting of all regular complex Borel measures on G , let $m^\wedge: G^\wedge \rightarrow \mathbb{C}$ denote its Fourier-Stieltjes transform given by

$$m^\wedge(z^\wedge) = \int_G \langle z, z^\wedge \rangle dm(z).$$

Let $M(G)^\wedge$ designate the space $\{m^\wedge: m \in M(G)\}$ (which is closed under products).

Recall that a measure $m \in M(G)$ is said to be continuous if $m(\{z\}) = 0$ for all $z \in G$ and it is said to be discrete if it is concentrated on a countable subset of G . It is shown in [2, pp. 70-71] that the linear functional L on $M(G)^\wedge$ given by $Lm^\wedge = m(\{0\})$ is positive. By means of Theorem 3, we now obtain a measure theoretic approach to Wiener's criteria for the continuity of a Borel measure on G (see [15], [7], and [16, p. 108]) which extends part of the result obtained in [2].

THEOREM 5. *Given a locally compact abelian group G with dual group $G^\wedge$, let V denote the real vector lattice, closed under products, generated by the space of all real-valued functions in $M(G)^\wedge$. Then, there exists a positive finitely additive set function $\mu: A \rightarrow [0,1]$ with the following properties.*

- 1) $\mu(G^\wedge) = 1$.
- 2) μ is inner and outer regular with respect to the zero and cozero sets (respectively) for the functions in V .
- 3) For all $m \in M(G)$ and all $z \in G$, we have

$$m(\{z\}) = \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} m^\wedge(z^\wedge) d\mu(z^\wedge)$$

and

$$\sum_{z \in G} |m(\{z\})|^2 = \int_{G^\wedge} |m^\wedge(z^\wedge)|^2 d\mu(z^\wedge)$$

and so m is continuous if and only if

$$\int_{G^\wedge} |m^\wedge(z^\wedge)|^2 d\mu(z^\wedge) = 0.$$

4) If $m_1, m_2 \in M(G)$ are such that $m_2^\wedge/m_1^\wedge$ is the uniform limit of A -measurable step functions and m_1 is a discrete measure for which $m_1^\wedge > 0$, then the bounded complex-valued function f on $G^\wedge$ given by

$$f(z) = \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} (m_2^\wedge(z^\wedge)/m_1^\wedge(z^\wedge)) d\mu(z^\wedge)$$

is such that

$$m_2(\{z\}) = \int_G f(z-w) dm_1(w)$$

for all $z \in G$.

PROOF. V is a vector lattice which is closed under products and contains the constant function 1. There exists a (unique) linear functional on V (which for ease of notation we also denote by L) whose restriction to the real-valued functions in $M(G)^\wedge$ is the given linear functional L . We can now apply Theorem 2 to obtain a positive finitely additive set function μ on A which is inner and outer regular with respect to the zero and cozero sets, respectively, for the functions in V , and which satisfies the equation

$$Lf = \int_{G^\wedge} f d\mu.$$

for all $f \in V$.

It is shown in [2, pp. 70-72] that the positive linear functional L has the following properties.

1) Any given real-valued (and so complex-valued) function in $M(G)^\wedge$ is the uniform limit of J_L -measurable step functions.

$$2) m_L(G^\bullet) = 1.$$

3) For all $m^\wedge \in M(G)^\wedge$ and all $z \in G$, we have

$$m(\{z\}) = \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} m^\wedge(z^\wedge) dm_L(z^\wedge)$$

and

$$\sum_{z \in G} |m(\{z\})|^2 = \int_{G^\wedge} |m^\wedge(z^\wedge)|^2 dm_L(z^\wedge),$$

and so, m is continuous if and only if

$$\int_{G^\wedge} |m^\wedge(z^\wedge)|^2 dm_L(z^\wedge) = 0.$$

Now $m_L = \mu$ on $J_L \cap A$ (see Step IX) and any given function in V is the uniform limit of $J_L \cap A$ -measurable step functions. So we get parts 1), 2) and 3) of the Theorem. We now prove part 4).

Since $m_2^\wedge/m_1^\wedge$ is the uniform limit of A -measurable step functions, the function f given by

$$f(z) = \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} (m_2^\wedge(z^\wedge)/m_1^\wedge(z^\wedge)) d\mu(z^\wedge)$$

exists for all $z \in G$. Hence

$$\begin{aligned} \int_G f(z-w) dm_1(w) &= \sum_{w \in G} f(z-w) m_1(\{w\}) \\ &= \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} \left[\sum_{w \in G} \langle w, z^\wedge \rangle m_1(\{w\}) \right] [m_2^\wedge(z^\wedge)/m_1^\wedge(z^\wedge)] d\mu(z^\wedge) \\ &= \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} m_1^\wedge(z^\wedge) [m_2^\wedge(z^\wedge)/m_1^\wedge(z^\wedge)] d\mu(z^\wedge) \\ &= \int_{G^\wedge} \overline{\langle z, z^\wedge \rangle} m_2^\wedge(z^\wedge) d\mu(z^\wedge) \\ &= m_2(\{z\}). \end{aligned}$$

This completes the proof.

REMARK 6. Note that $m_2^\wedge/m_1^\wedge$ is the uniform limit of A -measurable step functions whenever $m_1^\wedge$ is bounded away from 0 on $G^\wedge$. When G is the set of all integers (with discrete topology) and $m_1^\wedge > 0$ on $G^\wedge$ (i.e., on the unit circle in the complex plane), then $m_1^\wedge$ is bounded away from 0 and Wiener's lemma assures us that f is a discrete measure on G .

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