

ON A P-TYPE INTEGRAL

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Résumé

L'intégrale de Lebesgue fut généralisée par Titchmarsh [6], de l'école russe (Kolmogorov [2], etc.), et Kunugi [3]. Elle fut appelée respectivement (Q)-intégrale, (A)-intégrale et (ER)-intégrale. Dans un article antérieur [5], nous les avons étendues au cas de l'intégrale de Bochner. Dans cet article, nous les généralisons un peu plus (P-type intégrale) après avoir généralisé le théorème de Hahn-Banach et caractérisé l'espace des fonctions (ER)-intégrables comme la fermeture, dans notre espace des fonctions P-type intégrables, des fonctions intégrables de Bochner.

Abstract

Ordinary Lebesgue integral was generalized by Titchmarsh [6], Russian School (Kolmogorov [2], etc.) and Kunugi [3] called (Q)-integral, (A)-integral and (ER)-integral respectively. In a previous paper [5], we extended them to the case of Bochner integral. In this paper, we generalize them further (P-type integral) after generalizing Hahn-Banach theorem, and characterize the space of (ER)-integrable functions as a closure of the Bochner integrable functions in our space of P-type integrable functions.

0. Introduction

In 1929, E.C. Titchmarsh gave a definition of the (Q)-integral only by the limit (Q) in the following whenever it existed;

$$(Q) \quad I(f) = \lim_{n \rightarrow \infty} \int f^{(n)} dx$$

where $f^{(n)}$ was the truncated function of f by the number n ([6], p. 52). It was remarked ([6], p. 77, Theorem 12) that this (Q)-integration does not always satisfy the property $I(f+g) = I(f) + I(g)$. To satisfy the linearity of the (Q)-integral, the following condition (P) was already considered ([6], p. 79, Theorem 14);

$$(P) \quad \lim_{n \rightarrow \infty} n \cdot m\{s \in S ; |f(s)| \geq n\} = 0 .$$

Using both (Q) and (P) as axiom the (A)-integral was defined by Russian School (Kolmogorov, etc.) in 1933 [2]. On the other hand, Prof. K. Kunugi proposed the (ER)-integral by the method of ranked spaces in 1956 ([3]) and Prof. I. Amemiya and T. Ando proved that the (ER)-integrability coincided with the (A)-integrability in 1965 ([1]).

In this paper, we start with the condition (P) contrary to the Titchmarsh's case. In this position, the existence of integrals is not always guaranteed. We use the idea of the Banach limit and define our integral, called a P-type integral, to be linear and continuous operation. Then, the space (ER) which is a subspace of the space L^I of all P-type integrable functions is characterized as the closure w.r.t. a suitable quasi-norm $((\cdot))$ on L^I of the totality of simple (or step) functions or that of bounded integrable functions. Moreover, this P-type integral can be extended to the case where the range X of functions is a reflexive Banach space after generalizing the Hahn-Banach theorem.

Our discussions proceed in the following order: In §1, we prepare for some terminologies and notations. In §2 and §3, defining an admissible sequence

A , the spaces $K(A)$, K and a functional $||| \cdot |||$ on K , some properties of them are proved and in particular, it is shown that $||| f ||| = \|I(f)\|$ for any $f \in (ER)$. In §4, defining the space L^I of P -type integrable functions and the functional $((\cdot))$ on L^I , we show that $\overline{E} = \overline{B} = (ER)$ where the closure is taken with respect to $((\cdot))$. In §5, we extend I on (ER) to L^I using a generalized Hahn-Banach theorem also proved there. At last, in §6, we consider possible continuous linear mappings on (ER) and obtain the essential uniqueness which generalizes a result of Amemiya-Ando in the real valued case ([1], p. 138, Theorem 4).

1. Preliminaries on terminologies and notations ([1], [4], [5])

Let (S, S, μ) be a non-atomic, finite measure space with $\mu(S) = 1$ and X be a Banach space with the norm $\|\cdot\|$. By functions we mean those defined on S and having values in X . Let E be the set of all X -valued simple functions on S , i.e., every element $f \in E$ is denoted as $f = \sum_{i=1}^n x_i \mu(S_i)$ for some number n , some finite subset $\{x_1, \dots, x_n\}$ of X and some disjoint subfamily $\{S_1, \dots, S_n\}$ of S . Let L^0 be the set of all measurable functions on S , i.e., those for which there exist some sequences in E converging almost everywhere to them. Moreover, by B , we denote the set of all bounded measurable functions on S .

Let the truncated function of $f \in L^0$ by any positive number r be denoted as $f^{(r)}$, i.e.,

$$f^{(r)}(s) = \begin{cases} f(s) & \text{if } \|f(s)\| \leq r, \\ r \cdot \frac{f(s)}{\|f(s)\|} & \text{if } \|f(s)\| > r. \end{cases}$$

Any function $f \in L^0$ is said to be E.R. type integrable if, for the sequence $\{f^{(n)}\}$ of truncated functions, there exists the limit of the sequence $\left\{ \int_S f^{(n)} d\mu \right\}$ of their (Bochner) integrals with the following condition that

$$\mu\{\|f\| \geq n\} = o\left(\frac{1}{n}\right),$$

where $\{\|f\| \geq n\}$ is the simplified notation of the set $\{s; \|f(s)\| \geq n\}$. The above limit $\lim_{n \rightarrow \infty} \int_S f^{(n)} d\mu$ is said to be the E.R. type integral of the function f and denoted as $I(f)$. Let (ER) be the set of all E.R. type integrable functions.

2. Subset $K(A)$ of L^0

DEFINITION 1 (Admissible sequence A). Every sequence $A = \{A_n\}$ in S is said to be *admissible (sequence)* if the following conditions (i) and (ii) are satisfied:

- (i) $A_1 \supset A_2 \supset \dots \supset A_n \dots$
- (ii) $\lim_{n \rightarrow \infty} n \cdot \mu(A_n) = 0$ (i.e., $\mu(A_n) = o\left(\frac{1}{n}\right)$).

DEFINITION 2 ($K(A)$). For any admissible sequence A , let $K(A)$ be the set of all measurable functions satisfying the following condition:

$$\sup_n \frac{1}{n} \|f\|_{A_n} < +\infty \quad \text{where} \quad \|f\|_{A_n} = \text{ess sup}_{s \in A_n} \|f(s)\|$$

(i.e., $\|f\|_{A_n} = O(n)$).

EXAMPLE 1. For any $f \in (ER)$, there exists an admissible sequence A such that $f \in K(A)$. In fact, let $A_n = \{s \in S; \|f\| \geq n\}$, then $A = \{A_n\}$ is admissible.

EXAMPLE 2. $B \subset K(A)$ for any A ; in fact, $B = \bigcap_A K(A)$.

We may now prove some properties about the set $K(A)$.

THEOREM 1. $K(A)$ is a linear space.

PROOF. For any $f, g \in K(A)$, there exist constants $C \geq 0$ and $D \geq 0$ satisfying that $\|f\|_{A_n} \leq Cn$ and $\|g\|_{A_n} \leq Dn$ by the definition of $K(A)$. So we have, for any real number α, β ,

$$\|\alpha f + \beta g\|_{A_n} \leq |\alpha| \|f\|_{A_n} + |\beta| \|g\|_{A_n} \leq (|\alpha|C + |\beta|D)n ,$$

that is, $\|\alpha f + \beta g\|_{A_n} = O(n)$.

THEOREM 2. Let two admissible sequences be $A = \{A_n\}$ and $B = \{B_n\}$.

- (1) If $A_n \subset B_n$ for every n , then $K(A) \subset K(B)$.
 (2) If $f \in K(A) \cap K(B)$, then we have

$$\lim_{n \rightarrow \infty} \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus B_n} f \, d\mu \right\| = 0 .$$

PROOF. (1) From the following inequality

$$\|f\|_{B_n} = \operatorname{ess\,sup}_{S \setminus B_n} \|f\| \leq \operatorname{ess\,sup}_{S \setminus A_n} \|f\| = \|f\|_{A_n}$$

and the definition of $K(A)$ and $K(B)$, our conclusion follows.

- (2) For any $f \in K(A) \cap K(B)$, we have

$$\begin{aligned} \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus A_n \cup B_n} f \, d\mu \right\| &= \left\| \int_{B_n \setminus A_n} f \, d\mu \right\| \leq \int_{B_n \setminus A_n} \|f\| \, d\mu \\ &\leq \|f\|_{A_n} \cdot \mu(B_n) \leq c_1 \cdot n\mu(B_n) \end{aligned}$$

for some const. $c_1 > 0$ and, in the same way,

$$\begin{aligned} \left\| \int_{S \setminus B_n} f \, d\mu - \int_{S \setminus A_n \cup B_n} f \, d\mu \right\| &\leq \int_{A_n \setminus B_n} \|f\| \, d\mu \leq \|f\|_{B_n} \cdot \mu(A_n) \\ &\leq c_2 \cdot n\mu(A_n) \text{ for some const. } c_2 > 0 . \end{aligned}$$

Since the last terms of these estimations converge to 0 as $n \rightarrow \infty$, we get our conclusion.

COROLLARY. $K(A) + K(B) \subset K(A \cup B)$.

PROOF. By the Theorem 2(1), $K(A) \subset K(A \cup B)$ and $K(B) \subset K(A \cup B)$. Since $K(A \cup B)$ is a linear space (Th. 1), we have $K(A) + K(B) \subset K(A \cup B)$.

3. Functional $||| \cdot |||$ on $K = \bigcup_A K(A)$.

Let $K = \bigcup_A K(A) = \{f; \exists \text{ admissible sequence } A \text{ such that } f \in K(A)\}$.

K is a linear subspace of L^0 by the above mentioned properties.

DEFINITION 3 (Functional $||| \cdot |||$ on K). With every $f \in K$, we associate an extended real number $||| \cdot |||$ by

$$||| f ||| = \lim_n \sup \left\| \int_{S \setminus A_n} f \, d\mu \right\|$$

where $f \in K(A)$ and $A = \{A_n\}$. This number $||| f |||$ is uniquely defined without regard to any choice of admissible sequences A for which $f \in K(A)$ (Th. 2(2)).

THEOREM 3. The functional $||| \cdot |||$ on K is a semi-norm, i.e., satisfies the following (i) - (iii):

- (i) $f = 0$ a.e. $\Rightarrow ||| f ||| = 0$.
- (ii) For any $\alpha \in \mathbb{R}$, $||| \alpha f ||| = |\alpha| ||| f |||$.
- (iii) $||| f + g ||| \leq ||| f ||| + ||| g |||$.

These are easily shown from the definition of $||| \cdot |||$ and Th. 2.

THEOREM 4. If $f \in (ER)$, then $||| f ||| = \|I(f)\|$.

PROOF. Since the sequence $A = \{A_n\}$; $A_n = \{\|f\| \geq n\}$ is admissible by the definition of E.R. type integrability, we have $f \in K(A)$ and

$$\left\| \int_{S \setminus A_n} f \, d\mu - \int_S f^{(n)} \, d\mu \right\| \leq n \cdot \mu(A_n) \rightarrow 0$$

as $n \rightarrow \infty$. From the hypothesis, there exists $I(f) = \lim_{n \rightarrow \infty} \int_S f^{(n)} \, d\mu$. This implies the existence of $\lim_n \int_{S \setminus A_n} f \, d\mu$ and equalities

$$\lim_n \int_{S \setminus A_n} f \, d\mu = I(f)$$

and

$$|||f||| = \limsup_n \left\| \int_{S \setminus A_n} f \, d\mu \right\| = \|I(f)\|.$$

COROLLARY. For any $f \in (ER)$, $|||f||| = 0 \Leftrightarrow I(f) = 0$.

4. Subspace L^I of K and functional $((\cdot))$ on L^I

DEFINITION 4 (L^I). $L^I = \{f \in K ; |||f||| < +\infty\}$.

DEFINITION 5 ($((\cdot))$ on L^I). For every $f \in L^I$, associate the real value $((f))$ in the following way:

$$((f)) = \|f\|_0 + |||f||| \quad \text{where} \quad \|f\|_0 = \int_S \|f\| \wedge 1 \, d\mu.$$

By the properties of the functional $|||\cdot|||$ (Th. 3), L^I becomes a linear subspace of K . The functional $((\cdot))$ becomes a quasi-norm on L^I , that is, the following properties (1) - (4) are satisfied:

- (1) $((f)) = 0 \Leftrightarrow f = 0$ a.e.
- (2) $((f + g)) \leq ((f)) + ((g))$.
- (3) $((-f)) = ((f))$.
- (4) $a_n \rightarrow a$ in $\mathbb{R}$, $((f_n - f)) \rightarrow 0 \Rightarrow ((a_n f_n - af)) \rightarrow 0$.

In fact, (1) is shown since $((f)) = 0$ implies $\|f\|_0 = 0$ hence $f = 0$ a.e., and the other implication is obvious. (2) - (4) are easily shown since both $\|\cdot\|_0$ and $|||\cdot|||$ satisfy the same properties as (2) - (4). Consequently, L^I becomes a topological vector space by the quasi-norm $((\cdot))$. Let us remark that the L^0 -norm $\|\cdot\|_0$ gives the convergence in measure.

THEOREM 5. $\overline{E} = \overline{B} = (ER)$.

PROOF. We will prove in three steps.

(I) $\overline{E} = \overline{B}$.

The fact $E \subset B$ implies that $\overline{E} \subset \overline{B}$ is obvious. Conversely, let $f \in \overline{B}$. For any $\varepsilon > 0$, there exists a function $g \in B$ satisfying that

$((f - g)) < \varepsilon$. Since $g \in \mathcal{B}$, there exists some $h \in \mathcal{E}$ satisfying that

$((g - h)) < \varepsilon$. This implies that $((f - h)) < 2\varepsilon$.

(II) $f \in (\mathcal{E}\mathcal{R}) \Rightarrow f \in \overline{\mathcal{B}}$.

From the hypothesis, there exists an admissible sequence $\mathcal{A} = \{A_n\}$; $A_n = \{\|f\| \geq n \text{ and } f \in K(\mathcal{A})\}$. The truncated function $f^{(n)}$ of f is bounded and so E.R. type integrable. By the linearity of the set $(\mathcal{E}\mathcal{R})$ and Theorem 4, we have $|||f - f^{(n)}||| = \|I(f - f^{(n)})\|$. Since $I(f) = \lim_{n \rightarrow \infty} I(f^{(n)})$ and $I(f - f^{(n)}) = I(f) - I(f^{(n)})$, we have $|||f - f^{(n)}||| \rightarrow 0$ ($n \rightarrow +\infty$). Moreover, $f^{(n)} \rightarrow f$ a.e. hence $\|f - f^{(n)}\|_0 \rightarrow 0$ ($n \rightarrow +\infty$). Consequently, $f \in \overline{\mathcal{B}}$.

(III) $f \in \overline{\mathcal{B}} \Rightarrow f \in (\mathcal{E}\mathcal{R})$.

From the hypothesis, for any $\varepsilon > 0$, there exists some $g \in \mathcal{B}$ satisfying that $((f - g)) < \varepsilon$. Because $\overline{\mathcal{B}} \subset \overline{\mathcal{L}}^I \subset K$, there exists an admissible sequence $\mathcal{A} = \{A_n\}$ such that $f \in K(\mathcal{A})$. Since $g \in \mathcal{B} \subset K(\mathcal{A})$, we have $f - g \in K(\mathcal{A})$. So, we have

$$|||f - g||| = \lim_n \sup \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus A_n} g \, d\mu \right\| < \varepsilon,$$

that is,

$$\exists m; \quad \sup_{n \geq m} \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus A_n} g \, d\mu \right\| < \varepsilon.$$

By the boundedness of g ,

$$\operatorname{ess\,sup}_{s \in S} \|g(s)\| = M < +\infty.$$

From this and the definition of the admissible sequence,

$$\left\| \int_S g \, d\mu - \int_{S \setminus A_n} g \, d\mu \right\| \leq M\mu(A_n) \rightarrow 0 \quad (n \rightarrow +\infty),$$

i.e., there exists an ℓ satisfying that, for $n \geq \ell$,

$$\left\| \int_S g \, d\mu - \int_{S \setminus A_n} g \, d\mu \right\| < \varepsilon .$$

For this $n \geq \ell$,

$$\begin{aligned} \left\| \int_{S \setminus A_n} f \, d\mu - \int_S g \, d\mu \right\| &\leq \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus A_n} g \, d\mu \right\| + \left\| \int_{S \setminus A_n} g \, d\mu - \int_S g \, d\mu \right\| \\ &< 2\varepsilon . \end{aligned}$$

So, for $n, k \geq \ell$,

$$\begin{aligned} \left\| \int_{S \setminus A_n} f \, d\mu - \int_{S \setminus A_k} f \, d\mu \right\| &\leq \left\| \int_{S \setminus A_n} f \, d\mu - \int_S g \, d\mu \right\| + \left\| \int_S g \, d\mu - \int_{S \setminus A_k} f \, d\mu \right\| \\ &< 4\varepsilon , \end{aligned}$$

that is, there exists the limit $\lim_{n \rightarrow \infty} \int_{S \setminus A_n} f \, d\mu$. It follows that $f \in (ER)$. In fact, we have

$$\left\| \int_{S \setminus A_n} f \, d\mu - \int_S f^{(n)} \, d\mu \right\| \leq n\mu(A_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty .$$

5. Extension of I on (ER) to L^I

Having seen in previous sections, (ER) is a subspace of L^I , $\| \cdot \|$ on L^I is a semi-norm and the functional I on (ER) ; $I(f) = \lim_n \int_{S \setminus A_n} f \, d\mu$ ($= \lim_n \int_S f^{(n)} \, d\mu$) is linear and $\|I(f)\| = \|f\|$ on (ER) . Additionally, we

suppose that the Banach space X is reflexive. Then, applying the following generalized Hahn-Banach theorem, so to speak, there exists a linear operation $\tilde{I}$ on L^I extending I , i.e., satisfying

$$\tilde{I}(f) = I(f) \quad \text{for } f \in (ER)$$

and

$$\|\tilde{I}(f)\| \leq \|f\| \quad \text{for } f \in L^I .$$

The last representation means that the operation $\tilde{I} : L^I \rightarrow X$ is continuous on L^I by the semi-norm $||| \cdot |||$. By the definition of the quasi-norm $((\cdot))$, $\tilde{I}$ is continuous on L^I by the quasi-norm $((\cdot))$. Hence, we have obtained the following theorem.

THEOREM 6. Let L^I be the quasi-normed space with the quasi-norm $((\cdot))$ and I be the linear operation on the subspace (ER) on L^I defined by $I(f) = \lim_n \int_{S \setminus A_n} f \, d\mu$ for $f \in (ER)$. If the Banach space X is reflexive, then, there exists a linear continuous extension $\tilde{I}$ on L^I such that, for any $f \in (ER)$, we have

$$\tilde{I}(f) = \lim_n \int f^{(n)} \, d\mu$$

where $f^{(n)}$ is the truncated function of f by n .

COROLLARY. Under the same hypothesis as in the Theorem 6, $\tilde{I}$ on L^I is a linear continuous extension of $I : E \rightarrow X$ and $\tilde{I}(f) = \lim_n I(f_n)$ where $\{f_n\}$ is a sequence in E such that $f_n \rightarrow f$ by the quasi-norm $((\cdot))$.

The proof is obvious. Before giving an example which shows that (ER) is a proper subset of L^I , that is, $\tilde{I}$ is a proper extension of I , we prove the generalized Hahn-Banach theorem used for the above proof.

Generalized Hahn-Banach theorem. Suppose the following conditions

(1) - (4):

(1) X is a reflexive Banach space, i.e., $X \simeq X^{**}$,

(2) L is a linear space and M is a linear subspace of L ,

(3) $I : M \rightarrow X$ is a linear operation, and

(4) $p : L \otimes X^* \rightarrow \mathbb{R}$ is a semi-norm on the tensor product $L \otimes X^*$, that is, the following (i) - (iii) are satisfied; (i) $p(F) \geq 0$ for any $F \in L \otimes X^*$, (ii) $p(\alpha F) = |\alpha|p(F)$ for any $\alpha \in \mathbb{C}$ and $F \in L \otimes X^*$, (iii) $p(F + G) \leq p(F) + p(G)$ for any $F, G \in L \otimes X^*$, and p is strongly continuous with respect to $\phi \in X^*$ where $f \in L$ is arbitrarily fixed. Moreover the following inequality is

satisfied:

$$|\phi \circ I(h)| \leq p(h \otimes \phi)$$

for any $h \in M$ and any $\phi \in X^*$.

Then, there exists a linear extension $\tilde{I}$ of I on L such that $\tilde{I}(h) = I(h)$ for any $h \in M$ and $|\phi \circ \tilde{I}(f)| \leq p(f \otimes \phi)$ for any $f \in L$ and any $\phi \in X^*$.

PROOF. Define a linear functional $J : M \otimes X^* \rightarrow \mathbb{C}$ by the composition of I and ϕ ; $J(h \otimes \phi) = \phi \circ I(h)$. Then we have

$$|J(h \otimes \phi)| \leq p(h \otimes \phi).$$

By the Hahn-Banach extension theorem in the complex case, there exists a linear extension $\tilde{J}$ of J on $L \otimes X^*$, i.e.,

$$\tilde{J} : L \otimes X^* \rightarrow \mathbb{C},$$

$$\tilde{J}(h \otimes \phi) = J(h \otimes \phi) \quad \text{for } h \otimes \phi \in M \otimes X^*$$

and

$$|\tilde{J}(f \otimes \phi)| \leq p(f \otimes \phi) \quad \text{for } f \otimes \phi \in L \otimes X^*.$$

Define $\tilde{I} : L \rightarrow X^{**}$ by

$$\phi \circ \tilde{I}(f) = \phi(\tilde{I}(f)) = \tilde{I}(f)(\phi) = \tilde{J}(f \otimes \phi)$$

where $f \in L$ and $\phi \in X^*$. Since we have, for any fixed $f \in L$,

$$|\tilde{I}(f)(\phi)| = |\tilde{J}(f \otimes \phi)| \leq p(f \otimes \phi) \leq C\|f\|$$

for some constant $C > 0$, the linear operation $\tilde{I}(f)$ on X^* is continuous on X^* , i.e., $\tilde{I}(f) \in X^{**} \simeq X$. Moreover, for any $h \in M$, we have

$$\phi \circ \tilde{I}(h) = \tilde{J}(h \otimes \phi) = J(h \otimes \phi)$$

hence $\tilde{I}(h) = I(h)$.

Application. We put $L = L^I$, $M = (ER)$, $I(h) = \int_S h \, d\mu$ for $h \in (ER)$ and

$$p\left(\sum_{i=1}^m f_i \otimes \phi_i\right) = \lim_n \sup \left| \int_{S \setminus A_n} \sum_{i=1}^m \phi_i(f_i) \, d\mu \right|$$

for $f_i \in K(A)$ and $\phi_i \in X^*$ ($i = 1, 2, \dots, m$). $p\left(\sum_{i=1}^m f_i \otimes \phi_i\right)$ is well defined on $L \otimes X^*$ to be a semi-norm on $L \otimes X^*$ and satisfies

$$\begin{aligned} p(f \otimes \phi) &= \lim_n \sup \left| \int_{S \setminus A_n} \phi(f) \, d\mu \right| \\ &\leq \|\phi\| \cdot \lim_n \sup \left\| \int_{S \setminus A_n} f \, d\mu \right\| = \|\phi\| \cdot \|f\|. \end{aligned}$$

The above operation $I : M \rightarrow X$ satisfies that, for any $\phi \in X^*$,

$$\begin{aligned} |\phi \circ I(h)| &= \left| \int_S \phi(h) \, d\mu \right| = \lim_n \sup \left| \int_{S \setminus A_n} \phi(h) \, d\mu \right| \\ &= p(f \otimes \phi). \end{aligned}$$

By the preceeding extension theorem, there exists an extension $\tilde{I}$ of I on L satisfying that

$$\|\tilde{I}(f)\| = \sup_{\substack{\phi \in X^* \\ \|\phi\|=1}} |\phi \circ \tilde{I}(f)| \leq \sup_{\substack{\phi \in X^* \\ \|\phi\|=1}} p(f \otimes \phi) \leq \|f\|.$$

EXAMPLE. Take an $x \in X$; $\|x\| = 1$; $A = \{A_n\}$; $\mu(A_n) = 1/n^2$ so that $\lim_{n \rightarrow \infty} n \cdot \mu(A_n) = 0$ and

$$f(s) = \begin{cases} 2^{2^{4k+2}} \cdot x & (s \in A_{2^{2^{4k}} \setminus A_{2^{2^{4k+2}}}}) \\ -2^{2^{4(k+1)}} \cdot x & (s \in A_{2^{2^{4k+2}} \setminus A_{2^{2^{4(k+1)}}}}) \\ 0 & (\text{others}) \end{cases}$$

($k = 1, 2, \dots$) . It is shown that $f \in L^1$ but $f \notin (ER)$.

6. Continuous linear mappings on (ER)

Here we will consider the structure of continuous linear mappings on (ER) with values in X . At first, we introduce the following two notations:

$$(ER)_0 = \{f \in (ER) ; I(f) = 0\} ,$$

$$E_0 = \{g \in E ; I(g) = 0\} ,$$

hence, $\|(f)\|_0 = \|f\|_0 + \|I(f)\| = \|f\|_0 = \int_S \|f\| \wedge 1 \, d\mu$ for any $f \in (ER)_0$. Next, we suppose that (S, S, μ) is infinitely divisible in the following sense:

DEFINITION 6 (Infinitely divisible). (S, S, μ) is said to be *infinitely divisible* if, for any integer $N \geq 1$ and any $E \in S$, there exists a disjoint decomposition $\{E_1, E_2, \dots, E_N\}$ of E such that $\mu(E_i) = \frac{1}{N} \mu(E)$ ($i = 1, 2, \dots, N$) .

THEOREM 6. Let (S, S, μ) be infinitely divisible. If $J : (ER)_0 \rightarrow X$ is a continuous linear mapping, then we have $J \equiv 0$.

PROOF. For any $\varepsilon > 0$, put $B_X(0, \varepsilon) = \{x \in X ; \|x\| \leq \varepsilon\}$. If we prove that $J^{-1}B_X(0, \varepsilon) = (ER)_0$, then we have $J((ER)_0) \subset B_X(0, \varepsilon)$ for any $\varepsilon > 0$, so that $J((ER)_0) = \{0\}$, i.e., $J \equiv 0$ identically.

Let $V = J^{-1}B_X(0, \varepsilon) \subseteq (ER)_0$, then V is convex since $B_X(0, \varepsilon)$ is convex and J is linear. Because of continuity of J , there is a $\delta > 0$ satisfying that

$$B_{(ER)_0}(0, \delta) = \{f \in (ER)_0 ; \|f\|_0 \leq \delta\} \subset V .$$

For any $f \in (ER)_0$ and any $\varepsilon > 0$, there exists a $g \in E_0$ such that $\|f - g\|_0 < \varepsilon$. g is of the form $g = \sum_{i=1}^n \chi_{S_i} \cdot a_i$ where $\{S_1, \dots, S_n\} \subset S$ is a mutually disjoint set and $\{a_1, \dots, a_n\} \subset X$. Take an integer $N \geq 1$; $\frac{1}{N} < \delta$.

From the hypothesis, there exist disjoint decompositions $\{S_{ij} ; j = 1, 2, \dots, N\}$
 $(i = 1, 2, \dots, n)$, i.e.,

$$S_i = \bigcup_{j=1}^N S_{ij} , \quad \mu(S_{ij}) = \frac{1}{N} \mu(S_i) \quad \text{where } 1 \leq i \leq n \text{ and } 1 \leq j \leq N .$$

Defining $g_j = \sum_{i=1}^n N \chi_{S_{ij}} \cdot a_i$ ($1 \leq j \leq N$) , we have

$$I(g_j) = \sum_{i=1}^n N \cdot \mu(S_{ij}) \cdot a_i = \sum_{i=1}^n \mu(S_i) \cdot a_i = 0 ,$$

i.e., $g_j \in E_0$ ($1 \leq j \leq N$) and

$$\begin{aligned} \|g_j\|_0 &= \int_S \|g_j\| \wedge 1 \, d\mu \leq \sum_{i=1}^n \mu(S_{ij}) = \frac{1}{N} \sum_{i=1}^n \mu(S_i) \\ &\leq \frac{1}{N} < \delta , \end{aligned}$$

i.e., $g_j \in B_{(ER)_0}(0, \delta) \subset V$ ($1 \leq j \leq N$) . Since V is convex, we have

$$g = \frac{1}{N} \sum_{j=1}^N g_j \in V .$$

Moreover, since V is closed and $\varepsilon > 0$ is arbitral, we conclude

$$V = (ER)_0 .$$

THEOREM 7. Let (S, S, μ) be infinitely divisible. If $J : (ER) \rightarrow X$ is a continuous linear mapping, then there exists a continuous linear mapping $\theta : X \rightarrow X$ satisfying $J = \theta \circ I$ where I means the E.R. type integral.

PROOF. The E.R. type integral $I : (ER) \rightarrow X$ is always onto. In fact, for any $a \in X$, $\chi_S \cdot a \in E \subset (ER)$ and $I(\chi_S \cdot a) = a$. Let $\theta : X \rightarrow X$ be defined in the following way: For any $x \in X$, choose an $f \in (ER)$ such that $I(f) = x$ and let $\theta(x) = J(f)$. θ is well defined because, for another $g \in (ER)$ such that

$I(g) = x$, we have $I(f - g) = 0$, i.e., $f - g \in (ER)_0$ so $J(f - g) = 0$ by the previous Theorem 6, i.e., $J(f) = J(g)$. $\theta : X \rightarrow X$ is a linear mapping. In fact, for $x, y \in X$; $I(f) = x$, $I(g) = y$ and two scalars α and β , we have $I(\alpha f + \beta g) = \alpha x + \beta y$, then,

$$\begin{aligned}\theta(\alpha x + \beta y) &= J(\alpha f + \beta g) \\ &= \alpha J(f) + \beta J(g) = \alpha \theta(x) + \beta \theta(y) .\end{aligned}$$

$\theta : X \rightarrow X$ is continuous. In fact, for any sequence $\{x_n\}$ in X such that $x_n \rightarrow 0$ in X and any $\varepsilon : 0 < \varepsilon < 1$, there exists a number m satisfying that

$$n \geq m \Rightarrow \|x_n\| < \varepsilon < 1 .$$

Since we have

$$\|\chi_S \cdot x_n\|_0 = \int_S \|x_n\| \, d\mu < \varepsilon ,$$

i.e., $\chi_S \cdot x_n \rightarrow 0$ in (ER) and since J is continuous on (ER) , it follows that $J(\chi_S \cdot x_n) = \theta(x_n) \rightarrow 0$ in X .

Appendix

In a previous paper [5], it was erroneously stated that the semi-norm $\| \| h \| \|_3 = \| I(h) \|$ defined on the space E of simple (or step) functions can be extended automatically to a space of completion satisfying the condition (E) stated in that paper. Actually, this is not the case and this paper gives an answer to this extension problem. Namely, the semi-norm $\| \| h \| \|_3$ can be extended to the space $L^1 \subset K(A)$ by

$$\| \| f \| \| = \limsup_n \left\| \int_{S \setminus A_n} f \, d\mu \right\|$$

for any $f \in K(A)$ (above Theorem 4 and Definition 4). The completion $\widehat{E} \| \|_3$, as is seen from our Theorem 5, coincides with the space (ER) of functions

satisfying the conditions (P) and (Q) in the introduction which turns out to give another assurance to the maximality of the space (ER) stated in the paper by Amemiya-Ando ([1], Theorem 2).

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