

A NOTE ON THE HAHN-BANACH APPROXIMATION PROPERTY IN ORLICZ SPACES

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Résumé

Dans cette note, on démontre que dans la classe des espaces d'Orlicz des fonctions vectorielles $L^{\varphi}_{\mu}(E)$ avec φ vérifiant la condition $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$, la propriété d'approximation de Hahn-Banach est équivalente à la convexité locale.

An F -space X , i.e. a complete metric linear space, has the *Hahn-Banach approximation property* (HBAP) if each proper closed subspace of X is annihilated by some non-trivial continuous linear functional (cf. [5], [8], [9]) or, equivalently, if every weakly dense subspace is dense. If X is locally convex, then X clearly has the HBAP. A natural question is the following: If X has the HBAP, is then X locally convex? N. Kalton [5] showed a partially positive answer proving it for separable F -spaces with the point separation property. Shortly afterward J. Robert [9] gave a negative answer, i.e. in general the HBAP does not imply the local convexity (see also Ribe [8] and Kalton [6]). Furthermore it is known that for F -spaces the HBAP and the point separation property are in general independent properties (cf. [8], [9]).

Thus if X is a concrete (function or sequence) F -space it is interesting to know if the HBAP in X implies or not the local convexity and the point separation property. This has been studied for some F -spaces of analytic functions

in [11] and for the class of Orlicz spaces L^φ with $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} > 0$ by S. Cater [1] and Kalton [5]. We have analyzed in [4] these questions for Orlicz spaces of vector-valued functions $L_\mu^\varphi(E)$ with φ satisfying the condition $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$. There it is shown that, for a finite measure μ and a separable Banach space E , the HBAP in $L_\mu^\varphi(E)$ is equivalent to the local convexity.

The purpose of this note is to extend these results to larger classes of Orlicz spaces $L_\mu^\varphi(E)$ removing the condition of separability on E and considering an arbitrary σ -finite measure space.

Let (X, μ) be an arbitrary σ -finite measure space, $(E, \| \cdot \|)$ a Banach space over the field K ($\mathbb{R}$ or $\mathbb{C}$), and φ an Orlicz function, i.e. a non-zero nondecreasing function mapping $\mathbb{R}^+$ in $\mathbb{R}^+$, with $\varphi(0) = 0$, continuous on the left for $t > 0$ and continuous for $t = 0$. The Orlicz function space $L_\mu^\varphi(E)$ is the vector space of the strongly measurable E -valued functions f on X verifying that

$$\lambda_f(s, \varphi) = \int_X \varphi\left(\frac{|f|}{s}\right) d\mu < \infty \quad \text{for some } s > 0,$$

with functions agreeing almost everywhere identified. The space $L_\mu^\varphi(E)$ equipped with the F -norm $\|f\|_\varphi = \inf \{s > 0 : \lambda_f(s, \varphi) \leq s\}$ is an F -space. If $\varphi \in \Delta_2$, i.e. if there is $k > 1$ with $\varphi(2s) \leq k\varphi(s)$ for all $s > 0$, then $f \in L_\mu^\varphi(E)$ if and only if $\lambda_f(s, \varphi) < \infty$ for every $s > 0$. In particular when we take a bounded Orlicz function φ and a finite measure μ we have the space of all strongly measurable functions $L_\mu^0(E)$ with the topology of the convergence in measure.

If $\alpha = (\alpha_n)$ is a weight sequence (i.e. a positive reals sequence) then the Orlicz sequence space $\mathcal{L}_\alpha^\varphi(E)$ is the vector space consisting of all vector sequences $x = (x_n) \in E^{\mathbb{N}}$ such that $\lambda_x(s, \varphi) = \sum_{n=1}^{\infty} \varphi\left(\frac{|x_n|}{s}\right) \alpha_n < \infty$ for some $s > 0$ endowed with the F -norm $\|x\|_\varphi = \inf \{s > 0 : \lambda_x(s, \varphi) \leq s\}$. We refer to [7], [10], [12], [3] for conditions on φ under which the spaces $L_\mu^\varphi(K)$ and $\mathcal{L}_\mu^\varphi(K)$ are

(or are not) locally convex. It is easily seen that the Orlicz space of vector-valued functions $L_{\mu}^{\Phi}(E)$ is locally convex if and only if $L_{\mu}^{\Phi}(\mathbb{K})$ is locally convex.

We use later the next general observation:

PROPOSITION 1. *If an F-space X has the HBAP then every complemented subspace Y has it too.*

PROOF. Let H be an arbitrary closed subspace of Y . If Z is a subspace complementary to Y in X , let us consider the topological direct sum $H \oplus Z$ in X . As X has the HBAP there is a non-trivial continuous functional $f \in X'$ which is identically zero on $H \oplus Z$. In particular, f restricted to Z and H is zero too. Then the restriction of f to Y is a non-trivial continuous functional identically zero on H .

q.e.d.

PROPOSITION 2. *Let $\alpha = (\alpha_n)$ be an arbitrary weight sequence and $\varphi \in \Delta_2$. Then $\mathcal{L}_{\alpha}^{\Phi}(E)$ verifies the HBAP if and only if it is locally convex.*

PROOF. Let $x_0 \in E$ with unit norm, $E_0 \equiv \{\mathbb{K} \cdot x_0\}$ the one-dimensional subspace generated by x_0 and $P(x)$ the projection of E onto E_0 with norm $\|P\|$ ($< \infty$). Let us consider the subspace $H = \mathcal{L}_{\alpha}^{\Phi}(E_0)$ of $\mathcal{L}_{\alpha}^{\Phi}(E)$ consisting of the vector sequences $x = (x_n) \in E_0^N$ such that

$$\sum_{n=1}^{\infty} \varphi\left(\frac{|x_n|}{s}\right) \alpha_n < \infty \quad \text{for every } s > 0.$$

The operator $\bar{P}$ of $\mathcal{L}_{\alpha}^{\Phi}(E)$ defined by $\bar{P}(x) = (P(x_n))_{n=1}^{\infty}$ is a continuous projection of $\mathcal{L}_{\alpha}^{\Phi}(E)$ onto H . Indeed, since $P(x_n) \in E_0$ for each x_n of E we have that for every $s > 0$,

$$\sum_{n=1}^{\infty} \varphi\left(\frac{|P(x_n)|}{s}\right) \alpha_n \leq \sum_{n=1}^{\infty} \varphi\left(\frac{\|P\| |x_n|}{s}\right) \alpha_n < \infty$$

because $x \in \mathcal{L}_{\alpha}^{\Phi}(E)$ and $\varphi \in \Delta_2$. Thus $\bar{P}(x) \in H$ and for $\|x\|_{\varphi} \leq s$ we have that

$$\sum_{n=1}^{\infty} \left(\frac{|P(x_n)|}{\|P\|s} \right)_{\alpha_n} \leq \sum_{n=1}^{\infty} \left(\frac{|x_n|}{s} \right)_{\alpha_n} \leq s.$$

Hence $\|\bar{P}(x)\|_{\varphi} < \|P\| \|x\|_{\varphi}$ for every $x \in \mathcal{L}_{\alpha}^{\varphi}(E)$. It follows that H is a complemented subspace of $\mathcal{L}_{\alpha}^{\varphi}(E)$ and, by the above proposition, H verifies the HBAP.

Now the space $H \equiv \mathcal{L}_{\alpha}^{\varphi}(E_0)$ is separable because E_0 is separable too, and $\varphi \in \Delta_2$ (cf. [10]), and verifies the point separation property (the evaluation family separates points). Then we deduce, by Theorem 1 of Kalton [5], that $\mathcal{L}_{\alpha}^{\varphi}(E_0)$ and hence $\mathcal{L}_{\alpha}^{\varphi}(\mathbb{K})$ is locally convex. But this implies that the vector sequence space $\mathcal{L}_{\alpha}^{\varphi}(E)$ is locally convex too.

q.e.d.

In other words, the above proposition says that every non-locally convex weighted Orlicz sequence space $\mathcal{L}_{\alpha}^{\varphi}(E)$ (cf. [7], [2], [3]) has a non trivial quotient with trivial dual.

PROPOSITION 3. Let $\varphi \in \Delta_2$ and $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$. If $L_{\mu}^{\varphi}(E)$ has the HBAP, then it has the point separation property.

PROOF. By Proposition 2.1 of [4], it is sufficient to show that the measure μ is purely atomic. Otherwise, we suppose that there is an atomless subset B with strictly positive measure. Let us consider the subspace $L_{\mu}^{\varphi}(E, X \setminus B) = \{f \in L_{\mu}^{\varphi}(E) : f = f|_{X \setminus B}\}$, which is closed. Indeed, if $(f_n) \subset L_{\mu}^{\varphi}(E, X \setminus B)$ such that $f_n \rightarrow f$ in $L_{\mu}^{\varphi}(E)$ then $f_n \rightarrow f$ in $L_{\mu}^0(E)$. Now, as $f_n|_B = 0$ for every $n \in \mathbb{N}$ we have that

$$\mu\{x \in B : |f(x)| > s\} \leq \mu\{x \in X : |f_n(x) - f(x)| > s\},$$

which converges to zero when $n \rightarrow \infty$ for every $s > 0$. Hence $f \equiv 0$ on B , i.e.

$f \in L_{\mu}^{\varphi}(E, X \setminus B)$. Now, as $L_{\mu}^{\varphi}(E)$ is a topological direct sum of $L_{\mu}^{\varphi}(E, X \setminus B)$ and $L_{\mu}^{\varphi}(E, B)$ we have that every continuous functional on $L_{\mu}^{\varphi}(E)$ which annihilates $L_{\mu}^{\varphi}(E, X \setminus B)$ is trivial (cf. [4], lemma 1.2). So $L_{\mu}^{\varphi}(E)$ does not have the HBAP.

q.e.d.

The next result gives an extension of Theor. 2.3 in [4].

COROLLARY 4. If $\varphi \in \Delta_2$ and $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$ then $L_\mu^\varphi(E)$ has the HBAP if and only if it is locally convex.

PROOF. If $L_\mu^\varphi(E)$ verifies the HBAP, it follows from the above proposition and [4] (Prop. 2.1) that μ is purely atomic. Now, as μ is σ -finite, X has a decomposition $X = B \cup \left(\bigcup_{k=1}^{\infty} A_k \right)$ where (A_k) is a countable collection of pairwise disjoint atoms with $0 < \mu(A_n) = \alpha_n < \infty$ and $\mu(B) = 0$. Then, identifying $L_\mu^\varphi(E)$ with the weighted Orlicz sequence space $\mathcal{L}_\alpha^\varphi(E)$ we conclude from Proposition 2 that $L_\mu^\varphi(E)$ is locally convex.

q.e.d.

REMARK 1. In general a converse of Proposition 3 is not true, i.e. the point separation property in $L_\mu^\varphi(E)$ does not imply the HBAP when $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$. (Analogously it happens for $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} > 0$, [5].) Indeed, this follows from the Proposition 2.1 of [4] and [3] (Theor. 1). However, in the special case of a bounded function φ and a finite measure μ , the following statement is true. If $L_\mu^\varphi(E)$ has the point separation property then $L_\mu^\varphi(E) = L_\mu^0(E)$ is isomorphic to a product of countable many copies of E . So $L_\mu^0(E)$ is locally convex and has the HBAP.

There exists also unbounded Orlicz functions φ , verifying $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$, and finite measure spaces (X, μ) such that the point separation property in $L_\mu^\varphi(E)$ implies the HBAP, e.g.

Let $0 < p < 1$, $k \in \mathbb{N}$. If

$$\varphi(s) = \begin{cases} s^p & \text{if } s \in [0, 1] \\ (3 \cdot 5 \cdot \dots \cdot (2k-1))^{1-p} s^p & \text{if } s \in [(2k-1)!, (2k)!] \\ \frac{s}{(2 \cdot 4 \cdot \dots \cdot (2k))^{1-p}} & \text{if } s \in [(2k)!, (2k+1)!] \end{cases}$$

we have that $\varphi \in \Delta_2$, $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = 0$. By Theorem 2.1 of [13] there exists "weighted" sequences (α_n) with finite sum such that $\ell_\alpha^\varphi(E) = \ell^1(E)$. Now if $L_\mu^\varphi(E)$ has the point separation property, it follows from ([4], Proposition 2.1) that $L_\mu^\varphi(E) = \ell_\alpha^\varphi(E)$, so it is locally convex and has the HBAP. (Notice that this happens despite the fact that φ is not equivalent to any convex function near to ∞ .)

REMARK 2. When μ is a finite measure (or α is a weight sequence with $\lim_{n \rightarrow \infty} \alpha_n > 0$) we can consider, as it is usual, weaker Δ_2 -conditions in the above propositions. However, if we remove this Δ_2 -conditions we do not know if it remains true.

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