

ASYMPTOTIC EXPANSIONS OF LINEAR OPERATORS IN SOME VECTOR SPACES¹

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Introduction

This paper is a sequel to [3]. We study, in the abstract setting of $\text{Lin}(E^\infty)$, see [3], concepts of asymptotic expansions and of p -equivalence, establishing some mutual relationships; the main ideas are borrowed from [1].

1. We consider linear operators acting on E^∞ , the intersection of a convenient scale of Banach spaces E^s , $s \in \mathbb{R}$ (see [3]).

Let be $(r_j)_0^\infty$ a strictly decreasing sequence of real numbers, convergent to $-\infty$ if infinite, and let also be $(A_j)_0^\infty$ a sequence of operators in $\text{Lin}(E^\infty)$, such that $t \cdot o(A_j) = r_j$. We say that an operator $M \in \text{Lin}(E^\infty)$ has asymptotic expansion (A_j) — and write $M \sim \sum_{j=0}^\infty A_j$ — if the following sequence of (strict) inequalities holds (see [2]):

$$t \cdot o(M - \sum_{j=0}^N A_j) < r_N, \quad N = 0, 1, 2, \dots \quad (2.1)$$

The following is true:

PROPOSITION 1. If $M \sim \sum_{j=0}^\infty A_j$, then $t \cdot o(M) = t \cdot o(A_0) = r_0$.

First we get: $M = M - A_0 + A_0$, which implies

$$t \cdot o(M) \leq \sup\{t \cdot o(M - A_0), t \cdot o(A_0)\}.$$

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As $t \cdot o(M - A_0) < r_0$ and $t \cdot o(A_0) = r_0$, $\Rightarrow t \cdot o(M) \leq r_0$. Write now the equality: $A_0 = (A_0 - M) + M$; it results $t \cdot o(A_0) \leq \sup\{t \cdot o(A_0 - M), t \cdot o(M)\}$. If we assume by contradiction that $t \cdot o(M) < r_0$ strictly, we derive $t \cdot o(A_0) < r_0$ strictly; therefore the proposition is proved.

Let us remark now the following extension of Proposition 5 in [3]:

PROPOSITION 2. For any $L_1, L_2, \dots, L_n \in \text{Lin}(E^\infty)$ it is:

$$t \cdot o(L_1 + L_2 + \dots + L_n) \leq \sup\{t \cdot o(L_1), t \cdot o(L_2), \dots, t \cdot o(L_n)\}.$$

The proof is by induction on n , in a simple manner. Precisely:

$$\begin{aligned} t \cdot o(L_1 + L_2 + \dots + L_n + L_{n+1}) &\leq \sup\{t \cdot o(L_1 + \dots + L_n), t \cdot o(L_{n+1})\} \\ &\leq \sup\{\sup\{t \cdot o(L_1), \dots, t \cdot o(L_n)\}, t \cdot o(L_{n+1})\} \\ &\leq \sup\{t \cdot o(L_1) \dots t \cdot o(L_{n+1})\}. \end{aligned}$$

As a simple application, we obtain

PROPOSITION 3. Let be $A_0, A_1, \dots, A_n \in \text{Lin}(E^\infty)$ and $M = A_0 + A_1 + \dots + A_n$ where $t \cdot o(A_j) = r_j$, $r_0 > r_1 > \dots > r_n$. Then $M \sim \sum_{j=0}^n A_j$.

In fact, we have $M - A_0 = A_1 + \dots + A_n$ and

$$t \cdot o(M - A_0) \leq \sup\{t \cdot o(A_1), \dots, t \cdot o(A_n)\} = r_1 < r_0.$$

Also $t \cdot o(M - A_0 - A_1) = t \cdot o(A_2 + A_3 + \dots + A_n) \leq r_2 < r_1$, etc.

Next we shall follow closely ideas in ([1], pages 79-83), introducing the concept of p -equivalence of operators in $\text{Lin}(E^\infty)$, somehow related to the above studied asymptotic expansion. We work for simplicity in the subalgebra $\text{Lin}(E^\infty)_{\neq \emptyset}$, denoted here by E .

DEFINITION. Two operators $A, B \in E$, are said to be p -equivalent ($A \stackrel{p}{\sim} B$) iff $t \cdot o(A - B) \leq \inf(t \cdot o(A), t \cdot o(B))$, the inequality being strict if all the true orders are finite.

We remark that $A \overset{P}{\sim} A$, as $t \cdot o(\theta) = -\infty$; therefore the p -equivalence is a reflexive relation. Also, if $A \overset{P}{\sim} B$, then $B \overset{P}{\sim} A$ (as $t \cdot o(A - B) = t \cdot o(B - A)$). We give now

PROPOSITION 4. If $A, B \in E$ and $A \overset{P}{\sim} B$, then $t \cdot o(A) = t \cdot o(B)$.

In fact, from the equality $A = (A - B) + B$, we derive:
 $t \cdot o(A) \leq \sup\{t \cdot o(A - B), t \cdot o(B)\}$. As $t \cdot o(A - B) < t \cdot o(B)$, it follows that $t \cdot o(A) \leq t \cdot o(B)$. Using the above remarked symmetry of the p -equivalence, we find that $t \cdot o(B) \leq t \cdot o(A)$. We have also.

PROPOSITION 5. The p -equivalence has the transitivity property.

Let be $A \overset{P}{\sim} B$ and $B \overset{P}{\sim} C$; from the equality $A - C = (A - B) + (B - C)$, we find that $t \cdot o(A - C) \leq \sup\{t \cdot o(A - B), t \cdot o(B - C)\}$.

Also, from Prop. 4, we derive: $t \cdot o(A) = t \cdot o(B) = t \cdot o(C)$. We have two cases: if $t \cdot o(A - B) \leq t \cdot o(B - C)$, we see that $t \cdot o(A - C) \leq t \cdot o(B - C)$ which is $< t \cdot o(B)$ and $< t \cdot o(C)$ (with strict inequality in the finite case). Because of the equality of all three true orders, we find $A \overset{P}{\sim} C$. The same method works in the other case. We have also

PROPOSITION 6. Let be $A, B \in E$ and $t \cdot o(A - B) < t \cdot o(B)$ (strictly in the finite case). Then $A \overset{P}{\sim} B$.

Let us write again: $A = (A - B) + B$; hence
 $t \cdot o(A) \leq \sup\{t \cdot o(A - B), t \cdot o(B)\} = t \cdot o(B)$ and $t \cdot o(A) \leq t \cdot o(B)$. If this inequality is strict, from the decomposition $B = (B - A) + A$, we get
 $t \cdot o(B) \leq \sup\{t \cdot o(B - A), t \cdot o(A)\} = \sup\{t \cdot o(A - B), t \cdot o(A)\}$. Staying in the finite case (when $t \cdot o(A - B) > -\infty$), we find $t \cdot o(B) < t \cdot o(B)$ strictly, a contradiction. Hence, $t \cdot o(A) = t \cdot o(B)$ and $t \cdot o(A - B) < t \cdot o(A)$ too, so $A \overset{P}{\sim} B$. The case $t \cdot o(A - B) = -\infty$ is trivial.

Remark also that if $A \overset{P}{\sim} B$, then $-A \overset{P}{\sim} -B$, due to equality:
 $t \cdot o(-A + B) = t \cdot o(B - A) = t \cdot o(A - B) < \inf\{t \cdot o(A), t \cdot o(B)\}$ as requested.

Finally, we see that:

- i) $A \overset{p}{\sim} \theta \Leftrightarrow t \cdot o(A) = -\infty$,
- ii) if $t \cdot o(A) = t \cdot o(B) = -\infty$, then $A \overset{p}{\sim} B$ (as $t \cdot o(A - B) = -\infty$ too),
- iii) if $A \in E$ and $t \cdot o(B) = -\infty$, then $A + B \overset{p}{\sim} A$ (obvious).

2. In this paragraph, we establish some relations between concepts of asymptotic expansion and p -equivalence. First we state

PROPOSITION 7. Let be $M \in E$ and $(A_k)_0^\infty \subset E$ such that $t \cdot o(A_k) = r_k > -\infty$ and $M - (\sum_0^{k-1} A_j) \overset{p}{\sim} A_k$, $k = 1, 2, \dots$, $M \overset{p}{\sim} A_0$. Then the real sequence $(r_k)_0^\infty$ is strictly decreasing.

PROOF. From the equivalence $M - A_0 \overset{p}{\sim} A_1$, it follows that $t \cdot o(A_1) = t \cdot o(M - A_0)$. From the equivalence

$$M \overset{p}{\sim} A_0 \Rightarrow t \cdot o(M - A_0) < t \cdot o(A_0) = r_0;$$

therefore, $r_1 < r_0$. Let us assume to know already that $r_0 > r_1 > \dots > r_k$.

Then use the equivalence $M - (A_0 + A_1 + \dots + A_k) \overset{p}{\sim} A_{k+1}$ and derive that

$$t \cdot o(A_{k+1}) = t \cdot o(M - A_0 - A_1 - \dots - A_k).$$

Use also $M - (A_0 + \dots + A_{k-1}) \overset{p}{\sim} A_k$ and find that $t \cdot o(M - (A_0 + \dots + A_{k-1}) - A_k) < t \cdot o(A_k)$.

That is, we obtained: $r_{k+1} < r_k$, as requested. The following is an obvious

COROLLARY. Assume that $\lim_{k \rightarrow \infty} r_k = -\infty$. Then $M \sim \sum_0^\infty A_j$ (M has asymptotic expansion $\sum_0^\infty A_j$).

We have also the

PROPOSITION 8. Let be, in E , $M \sim \sum_0^\infty A_k$, where $\{t \cdot o(A_k)\}_0^\infty$ is a strictly decreasing sequence. Then $M - (\sum_0^{k-1} A_j) \overset{p}{\sim} A_k$, $k = 1, 2, \dots$, and $M \overset{p}{\sim} A_0$.

The proof is a consequence of Proposition 6. In fact, from the inequality:

$$t \cdot o(M - A_0) < t \cdot o(A_0) \text{ one deduces: } M \overset{p}{\sim} A_0.$$

$$t \cdot o(M - A_0 - A_1) < t \cdot o(A_1) \text{ we get } M - A_0 \overset{p}{\sim} A_1; \text{ from}$$

$t \cdot o(M - A_0 - A_1 - \dots - A_k) < t \cdot o(A_k)$, we find that

$$M - A_0 - A_1 - \dots - A_{k-1} \overset{p}{\sim} A_k. \quad \text{Q.E.D.}$$

We shall now establish an uniqueness result for asymptotic expansions, modulo operators of true order $= \infty$. Precisely, the following is true:

THEOREM 1. Let be $F \subset E$ having the property that: $A, B \in F$ and $A \overset{p}{\sim} B \Rightarrow t \cdot o(A - B) = \infty$. Assume that $M \in E$ has two asymptotic expansions: $M \sim \sum_0^\infty A_j$ and $M \sim \sum_0^\infty B_j$ where $A_j, B_j \in F$, $j = 0, 1, \dots$. Then $t \cdot o(A_j) = t \cdot o(B_j)$, $j = 0, 1, \dots$ and $A_j - B_j \in L_\infty$, $j = 0, 1, 2, \dots$.

PROOF. First we use Prop. 8 and find: $M \overset{p}{\sim} A_0$ and $M \overset{p}{\sim} B_0$; therefore $A_0 \overset{p}{\sim} B_0$. Next, again by the above proposition, $M - A_0 \overset{p}{\sim} A_1$ and $M - B_0 \overset{p}{\sim} B_1$. Remark now that $M - A_0 \overset{p}{\sim} M - B_0$ as their difference is $B_0 - A_0 \in L_\infty$. Thus $A_1 \overset{p}{\sim} B_1$. For arbitrary n we use induction; assume the statement true for $j = 0, 1, \dots, n$. Then use the relations $M - (\sum_0^n A_j) \overset{p}{\sim} A_{n+1}$, $M - (\sum_0^n B_j) \overset{p}{\sim} B_{n+1}$. Again, the difference in the left-hand side is $\sum_0^n (B_j - A_j)$ which belongs to L_∞ . Q.E.D.

Remark also

PROPOSITION 9. Assume that M, N belong to E and $M \sim \sum_0^\infty A_k$, $N \sim \sum_0^\infty A_k$, $(A_k)_0^\infty \subset E$ (so that $t \cdot o(A_k) \downarrow -\infty$). Then $t \cdot o(M - N) = -\infty$.

In fact, from the definition of asymptotic expansion, we get:

$t \cdot o(M - (A_0 + \dots + A_n)) < t \cdot o(A_n)$, $t \cdot o(N - (A_0 + \dots + A_n)) < t \cdot o(A_n)$. Hence, $t \cdot o(M - N) < t \cdot o(A_n)$, $\forall n = 1, 2, \dots$. As $n \rightarrow \infty$, we get the result.

PROPOSITION 10. Let be in E , $M \sim \sum_0^\infty A_k$ and $0 \neq \lambda \in K$ (the complex field for the E^S). Then $\lambda M \sim \sum_0^\infty \lambda A_k$.

This follows from the equality: $\lambda M - \sum_0^N \lambda A_k = \lambda(M - \sum_0^N A_k)$ and from Prop. 3 in [3].

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