

PARTIAL MIXING FLOWS

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Résumé

Le but de cette note est de construire, par une modification de la construction connue pour les transformations partiellement mélangeantes, des flots partiellement mélangeants. On obtient des flots α -mélangeants pour tout α ($0 < \alpha < 1$). En fait, pour α fixe, on obtient, par moyens des suites "éventuellement indépendantes", une infinité non dénombrable de flots α -mélangeants distincts. Chaque flot construit ici est d'entropie zéro.

1. Introduction

Partial mixing transformations were introduced in [3] and a continuum of non-isomorphic partial mixing transformations with zero entropy were constructed in [4]. Our purpose is to make analogous constructions for flows. Preliminary definitions are given in section 2. A relatively simple construction of a partial mixing flow is given in section 3. In section 4 we construct, for each $\alpha \in (0,1)$, an α -mixing flow. In section 5 we fix α and use eventually independent sequences [2] to show the existence of uncountably many pair-wise non-isomorphic α -mixing flows. All flows constructed have zero entropy. The constructions use ideas from [2, 3, 4, 5].

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2. Preliminaries

Let $(X, \mathcal{B}, m)$ be a probability space isomorphic to the unit interval with Lebesgue measure. A measure preserving flow $(T_t : -\infty < t < \infty)$ defined on X is *partial mixing* if there exists $\beta > 0$ such that

$$(2.1) \quad \liminf_{t \rightarrow \infty} m(T_t A \cap B) \geq \beta m(A)m(B), \quad A, B \in \mathcal{B}.$$

A partial mixing flow is α -mixing if (2.1) holds for $\beta = \alpha$ but (2.1) does not hold for $\beta = \alpha + \epsilon$, $\epsilon > 0$.

All flows constructed will be flows built under a step function with an interval base. Thus the measure space will be a subset of G of the plane consisting of rectangles with normalized Lebesgue measure. A transformation T is defined on the interval base. Under the flow, a point (x, y) moves upward at unit speed until it reaches the top of G and then jumps to the base at $T(x)$. Thus if $h = h(x, y)$ denotes the height from (x, y) to the top of G , then $T_t(x, y) = (x, y+t)$, $0 \leq t < h$, and $T_h(x, y) = (T(x), 0)$.

All rectangles R will be left-bottom-closed and right-top-open. A *tower* G is an ordered set of disjoint rectangles; hence

$$G = (R_1, R_2, \dots, R_k).$$

The base of a rectangle R is an interval $[a, b]$, which we denote by $\underline{R}$. The width of R is $w(R) = b - a$. The base of G is $\underline{G} = \bigcup_{i=1}^k \underline{R}_i$ and the width of G is $w(G) = \sum_{i=1}^k w(R_i)$. Given two towers G_1 and G_2 , we let (G_1, G_2) denote the tower consisting of the rectangles of G_1 followed by the rectangles of G_2 .

Let R be a rectangle and $0 < p < 1$. A subrectangle pR is a rectangle whose base is a subinterval of $\underline{R}$ with $w(pR) = pw(R)$ and the same height as R . Hence $m(pR) = pm(R)$, where m denotes planar Lebesgue measure. Given a positive integer $k \geq 2$, it is clear we can divide a rectangle R into k disjoint rectangles $R_j = R/k$, $1 \leq j \leq k$.

Given $G = (R_j : 1 \leq j \leq k)$ and $0 < p < 1$, define the *subtower* pG as

$$pG = (pR_j : 1 \leq j \leq k) .$$

We refer to pG as a p -size copy of G , or simply a copy of G . Setting $p_j = w(R_j)/w(G)$, $1 \leq j \leq k$, we can decompose G into two $\frac{1}{2}$ -size copies G_0 and G_1 and then decompose G_1 further into k copies $G_1^1, \dots, G_1^k$ where G_1^j is a p_j -size copy of G_1 , $1 \leq j \leq k$. The width of G_1^j is then equal to the width of the j^{th} rectangle in G_0 , so we can form a new tower $S(G)$ by placing each G_1^j above the j^{th} rectangle in G_0 . Thus $S(G)$ consists of a $\frac{1}{2}$ -size copy G_0 of G and a copy of G above each rectangle in G_0 .

Given a point (x, y) in a tower G , let k denote the height of (x, y) above $\underline{G}$, and let h denote the height of the top of the rectangle of G containing (x, y) above (x, y) . We define $T(G)_t(x, y) = (x, y+t)$ for $-k \leq t \leq h$. Noting that $T(S(G))_t$ extends $T(G)_t$, we obtain a measure-preserving flow $T(G)$ on G by setting, for $t \in \mathbb{R}$ and $(x, y) \in G$, $T_t(G)(x, y) = \lim_{k \rightarrow \infty} T(S^k(G))_t(x, y)$ which limit is easily seen to exist.

A rectangle A of height ϵ is called an ϵ -rectangle in G if A is contained in a rectangle R in G and $w(A) = w(R)$. That is, A is a strip of height ϵ in R with the same width as R . If the heights of the rectangles in G are not rationally related, then we refer to G as an M -tower. It will now be shown that if G is an M -tower, then $T_t(G)$ is mixing on ϵ -rectangles in G .

LEMMA 2.2. If G is an M -tower, $\epsilon > 0$ is less than the minimum height of the rectangles in G , and A and B are ϵ -rectangles in G , then

$$\lim_{t \rightarrow \infty} m(T_t(G)(A) \cap B) = m(A)m(B)/m(G) .$$

PROOF. The definition of the S operator implies $S(G)$ consists of a bottom copy G_0 of G and a copy of G above each rectangle in G_0 . We refer to G_0 as the level 1 copy of G in $S(G)$ and the copies of G above the rectangles in G_0 as the level 2 copies of G in $S(G)$. In general, for $k \geq 1$, $S^k(G)$ consists

of a bottom copy $S^k(G)_0$ of $S^{k-1}(G)$ and a copy $S^k(G)_{1,j}$ of $S^{k-1}(G)$ above each rectangle R_j in $S^k(G)_0$. We refer to the level i copies of G in $S^k(G)_0$ as level i copies of G in $S^k(G)$, and we refer to the level i copies of G in the $S^k(G)_{1,j}$ as level $i + 2^{k-1}$ copies of G in $S^k(G)$.

Now let G consist of q rectangles R_j of height h_j , $1 \leq j \leq q$. Let D_j be the ϵ -rectangle in R_j whose base coincides with the base of R_j . Thus $D = \bigcup_{j=1}^q D_j$ is a strip of height ϵ on the base of G . Fix an ϵ -rectangle B in G and define

$$(2.3) \quad z(t) = m(T_t(G)D \cap B)/m(D) .$$

It will be shown that $z(t)$ satisfies

$$(2.4) \quad z(t) = \sum_{j=1}^q m(D_j) z(t - h_j) / m(D) .$$

To verify (2.4), we shall consider $T_t(G)$ replaced by $T(S^k(G))_t$ for k very large compared to t . The picture of $S^k(G)$ consisting of level i copies of G above rectangles in level $i - 1$ copies of G , $1 \leq i \leq 2^k$, implies that an ϵ -rectangle D_j appears as ϵ -rectangles in these copies of G in $S^k(G)$. Consider one such ϵ -rectangle D_* in a level i copy of G , $i < 2^k$. Now D_* is in a rectangle in a level i copy of G and there is a level $i + 1$ copy of G above this rectangle. Since D_j is on the base of R_j , D_* will be on the base of a rectangle in a level i copy of G of height h_j . Thus $T(S^k(G))_{h_j} D_*$ will have a base equal to the base of the level $i + 1$ copy of G above D_* . Now because of the way in which the level $i + 1$ copies of G are stacked, if $T(S^k(G))_t D_*$ is defined in $S^k(G)$, then $T(S^k(G))_t D_*$ intersects B in the same proportion as the base D intersects B , except delayed by h_j ; hence

$$(2.5) \quad m(T(S^k(G))_t D_* \cap B) / m(D_*) = z(t - h_j) ,$$

or

$$(2.6) \quad m(T(S^k(G))_t D_* \cap B) = z(t - h_j) m(D_*) .$$

Let D^* denote the union of the sets D_* for which $T(S^k(G))_t D_*$ is defined in $S^k(G)$; hence (2.6) implies

$$(2.7) \quad m(T(S^k(G))_t D^* \cap B) = z(t - h_j)m(D^*) .$$

Given $\eta > 0$, one can choose k so large relative to t so that $m(D_j - D^*) < \eta$. Since η is arbitrary, we can replace D^* by D_j in (2.7) and $T(S^k(G))_t$ is replaced by its limit $T_t(G)$. Thus we have

$$(2.8) \quad m(T_t(G)D_j \cap B) = z(t - h_j)m(D_j) .$$

Since $D = \bigcup_{j=1}^q D_j$, (2.8) implies

$$(2.9) \quad m(T_t(G)D \cap B) = \sum_{j=1}^q z(t - h_j)m(D_j) .$$

Dividing (2.9) by $m(D)$ implies (2.4). Since G is an M -tower, (2.4) and the Renewal Theorem imply $\lim_{t \rightarrow \infty} z(t)$ exists. The ergodic theorem for flows now yields $\lim_{t \rightarrow \infty} z(t) = m(B)/m(G)$.

From (2.8) we now obtain

$$(2.10) \quad \lim_{t \rightarrow \infty} m(T_t(G)D_j \cap B) = m(D_j)m(B)/m(G) .$$

If A is an ϵ -rectangle in G , then A is an ϵ -strip in a rectangle R_j in G . Let h be the height of the base of A ; hence $A = T_h(G)D_j$. From (2.10) we conclude

$$(2.11) \quad \lim_{t \rightarrow \infty} m(T_t(G)A \cap B) = m(A)m(B)/m(G) .$$

Thus $T_t(G)$ is mixing on ϵ -rectangles.

Given $\epsilon > 0$, Lemma 2.2 implies there exists $N(G, \epsilon)$ such that $t \geq N(G, \epsilon)$ implies $m(T_t(G)A \cap B) \geq m(A)m(B)/2m(G)$. We will say that *mixing starts in G for ϵ -rectangles at $N(G, \epsilon)$* .

The proof of the following simple lemma is omitted.

LEMMA 2.12. Let $N > N(G, \epsilon)$. There exists $k = k(G, \epsilon, N)$ such that if T_t extends $T(S^k(G))_t$, then

$$m(T_t A \cap B) \geq m(A)m(B)/2m(G), \quad N(G, \epsilon) \leq t \leq N,$$

where A and B are ϵ -rectangles in G .

Suppose a tower $G = (R_1, R_2, \dots, R_k)$ and $\lambda > 0$ satisfy $w(R_i) = n_i \lambda$, with $n_i \in \mathbb{Z}$, $1 \leq k \leq i$. Then we can subdivide each R_i into $n_i \frac{1}{n_i}$ -size copies of R_i and stack all the resulting rectangles consecutively to obtain a single rectangle of width λ . We denote this rectangle by $V(R)$. For definiteness, we may assume that λ is the largest number with the above property.

If R is a rectangle of height h and n is a positive integer, we let $S_n(R)$ denote the rectangle obtained by subdividing R into $n \frac{1}{n}$ -size copies of R and stacking them consecutively. We will use the operations V and S_n to prevent mixing in the constructions to follow. Note that if the rectangle R has height h and A is an ϵ -rectangle in R , and T_t extends $T(S_n(R))_t$, then

$$(2.13) \quad m(T_n A \cap A) \geq (1 - \frac{1}{n})m(A).$$

Similarly, if G is a tower as above, A is an ϵ -rectangle in G , h is the height of $V(G)$, and it T_t extends $T(S_n(V(G)))_t$, then again (2.13) holds.

Suppose A is an ϵ -rectangle in a tower G , and G_1 is a tower obtained from G by any combination of the above operations on towers, such that a fraction α of A appears in G_1 as a union of ϵ -rectangles. Then we say that αA appears in (or is contained in) G_1 .

3. Partial Mixing

We shall now construct a partial mixing flow with $\beta = 1/8$ in (2.1). An informal description of the construction will be presented first and then the de-

tails will be given. At each stage in the construction a varying set of measure $\frac{1}{4}$ will be mixed. Also, the use of the operations V and S_2 at each stage will force a fixed set A with $m(A) < \frac{1}{4}$ to intersect itself infinitely often in a set with measure exceeding $m(A)/4$. This guarantees that the flow is not mixing.

We start with an M -tower G_0 with $m(G_0) = 1$, and with the widths of the rectangles of G_0 being rationally related. (In fact, we may as well take all the widths to be equal.) Decompose G_0 into two equal copies $G_1 = .5G_0$ and $G'_1 = .5G_0$. Let $R_1 = V(G'_1)$. Note that $m(G_1) = m(R_1) = \frac{1}{2}$ and if A is a rectangle in G_0 , then $A/2$ is contained in G_1 and R_1 . We shall first form an M -tower $G_2 = (.5G_1, .5R_1)$ for the second stage. There are $.5G_1$ and $.5R_1$ remaining, each of measure $\frac{1}{4}$.

We now mix the remaining copy $.5G_1$ of measure $\frac{1}{4}$. Choose a positive integer k_1 and mix $.5G_1$ by forming $G_{1,1} = S^{k_1}(.5G_1)$. There are two criteria for choosing k_1 . First k_1 must be so large that certain rectangles in $.5G_1$ start to mix. Second, k_1 must be so large that this mixing continues in $.5G_1$ until mixing starts in $.5G_2$. Note that mixing starts in $.5G_2$ at the same time as it would start in G_2 since $.5G_2$ is a copy of G_2 . Also G_2 is formed before k_1 is chosen.

Since G_1 contains $A/2$ if A is a rectangle in G_0 , the mixing of $.5G_1$ implies mixing of $A/4$ for certain rectangles A in G_0 . Another $A/4$ will continue to mix in $.5G_2$.

After $G_{1,1}$ is formed, it is combined with the remaining $.5R_1$ to obtain the rectangle $R_{1,1} = V((G_{1,1}, .5R_1))$. We then form $R_2 = S_2(R_{1,1})$. Now G_2 is an M -tower and R_2 is a rectangle with $m(G_2) = m(R_2) = \frac{1}{2}$. If A is a rectangle in either G_0 or G_1 , then G_2 and R_2 each contain $A/2$.

The details of the first stage will now be given. Let $\epsilon_1 > 0$ be less than the minimum height of the rectangles of G_0 , and for $m \geq 2$ let $\epsilon_m = \epsilon_1/2^{m-1}$. If A is an ϵ_1 -rectangle in G_0 , then $A/4$ is in $.5G_1$. Since $.5G_1$ is a

copy of G_1 , ϵ_1 -rectangles in $.5G_1$ start to mix at $N_1 = N(G_1, \epsilon_1)$. Also ϵ_2 -rectangles in $.5G_2$ start to mix at $N(G_2, \epsilon_2)$. We want ϵ_1 -rectangles in $.5G_1$ to continue to mix until ϵ_2 -rectangles in $.5G_2$ start to mix. Choose $N_2 > \max(N_1, N(G_2, \epsilon_2))$ and let $k_1 = k(G_1, \epsilon_1, N_2)$. Denote $G_{1,1} = S^{k_1}_{\epsilon_1}(.5G_1)$. Thus if T_t extends $T(G_{1,1})_t$ and E and F are ϵ_1 -rectangles in $.5G_1$, then

$$(3.1) \quad m(T_t E \cap F) \geq m(E)m(F)/2m(G_{1,1}), \quad N_1 \leq t \leq N_2.$$

If A and B are ϵ_1 -rectangles in G_0 , then $E = A/4$ and $F = B/4$ are ϵ_1 -rectangles in $.5G_1$. Since $m(G_{1,1}) = \frac{1}{4}$, (3.1) implies

$$(3.2) \quad m(T_t A \cap B) \geq m(A)m(B)/8, \quad N_1 \leq t \leq N_2.$$

Let $R_{1,1} = V((G_{1,1}, .5R_1))$ and $R_2 = S_2(R_{1,1})$.

If A is a rectangle in G_0 , then $A/2$ is in $R_{1,1}$. Let h_1 be the height of $R_{1,1}$. The formation of R_2 and (2.13) imply that if T_t extends $T(R_2)_t$, then

$$(3.3) \quad m(T_{h_1} A \cap A) \geq m(A)/4.$$

The general induction step in the construction will now be described. At the n^{th} stage of the construction we have M towers G_i and rectangles R_i , $1 \leq i \leq n$, each of measure $\frac{1}{2}$, and increasing sequences of integers $\{N_i\}_{i=1}^n$ and $\{k_i\}_{i=1}^{n-1}$ satisfying $N_{i+1} > \max(N_i, N(G_{i+1}, \epsilon_{i+1}))$ and $k_i = k(G_i, \epsilon_i, N_{i+1})$. In addition, for $1 \leq i \leq n-1$, we have M -towers $G_{i,i} = S^{k_i}_{\epsilon_i}(.5G_i)$ such that if T_t extends $T(G_{i,i})_t$ and E and F are ϵ_i -rectangles in $.5G_i$, then

$$(3.4) \quad m(T_t E \cap F) \geq m(E)m(F)/2m(G_{i,i}), \quad N_i \leq t \leq N_{i+1}.$$

Let $G_{n+1} = (.5G_n, .5R_n)$. Choose $N_{n+1} > \max(N_n, N(G_{n+1}, \epsilon_{n+1}))$. Let $k_n = k(G_n, \epsilon_n, N_{n+1})$ and $G_{n,n} = S^{k_n}_{\epsilon_n}(.5G_n)$. Thus, if E and F are ϵ_n -rectangles in $.5G_n$ and T_t extends $T(G_{n,n})_t$, then

$$(3.5) \quad m(T_t E \cap F) \geq m(E)m(F)/2m(G_{n,n}), \quad N_n \leq t \leq N_{n+1}.$$

If A and B are ε_i -rectangles, $1 \leq i \leq n$, then $A/4$ and $B/4$ are disjoint unions of ε_n -rectangles in $.5G_n$. Hence (3.5) implies

$$(3.6) \quad m(T_t A \cap B) \geq m(A)m(B)/8, \quad N_n \leq t \leq N_{n+1}.$$

Let $R_{1,n} = V((G_{n,n}, .5R_n))$ and $R_{n+1} = S_2(R_{1,n})$. If A is a rectangle in R_1 , $1 \leq i \leq n$, then $A/2$ is in $R_{1,n}$. Let h_n be the height of $R_{1,n}$. The formation of R_{n+1} and (2.13) imply that if T_t extends $T(R_{n+1})_t$, then

$$(3.7) \quad m(T_{h_n} A \cap A) \geq m(A)/4.$$

It is easily seen that the class of ε_n -rectangles in G_n and R_n , $n \geq 1$, generate the full σ -algebra $\mathcal{B}$. Hence an approximation argument and (3.6) imply that if A and B are measurable sets, then

$$(3.8) \quad \liminf_{t \rightarrow \infty} m(T_t A \cap B) \geq m(A)m(B)/8.$$

Thus (2.1) is satisfied for $\beta = 1/8$.

To see that the flow is not mixing, let A be a rectangle in R_1 of measure less than $\frac{1}{4}$. Since $h_n \rightarrow \infty$, (3.7) implies that the flow cannot be mixing.

4. Non-isomorphic Partial Mixing Flows

We will construct an α -mixing flow for $0 < \alpha < 1$. Since α -mixing flows are not isomorphic for distinct α , we will obtain a continuum of non-isomorphic partial mixing flows. The flows will also have zero entropy.

An α -mixing flow is obtained by mixing an α part of the space and unmixing a $1 - \alpha$ part of the space at each stage in the construction. The $1 - \alpha$ part is then mixed back in with the α part before proceeding to the next stage. This requires some additional definitions.

Let G be a tower with q rectangles and let $w_j(G)$ and $h_j(G)$ denote the width and height of the j^{th} rectangle, $1 \leq j \leq q$. Two towers G_1 and G_2

with q columns are *similar* if there exists $a > 0$ such that $w_j(G_1) = aw(G_2)$ and $h_j(G_1) = h_j(G_2)$, $1 \leq j \leq q$. A copy of a tower G is similar to G . However, a tower similar to G may not be a copy of G since it may not be contained in G .

Let G be as above and let R be a rectangle disjoint from G with height h and width w . Let $H = \min\{h_j(G) : 1 \leq j \leq q\}$ and let K be the largest integer such that $Kh < H$. Assume that $K > 0$. We shall use R to form a tower $G(R)$ similar to G . It may be necessary to use an additional amount of measure μ , where $\mu \leq hw/K$. Let n_j denote the largest positive integer such that $n_j h \leq h_j(G)$, $1 \leq j \leq q$. Thus $n_j \geq K$, $1 \leq j \leq q$. Define v_j as

$$(4.1) \quad v_j = ww_j(G) / \sum_{i=1}^q n_i w_i(G), \quad 1 \leq j \leq q.$$

From (4.1) we obtain

$$(4.2) \quad \sum_{j=1}^q n_j v_j = w,$$

$$(4.3) \quad v_i/v_j = w_i(G)/w_j(G), \quad 1 \leq i, j \leq q.$$

By (4.2) we can divide R into $\sum_{j=1}^q n_j$ rectangles, where n_j rectangles have width v_j , $1 \leq j \leq q$. Stack the rectangles of width v_j to form one rectangle of height $n_j h$. If $n_j h < h_j(G)$ then add an additional rectangle of height $h_j(G) - n_j h$ and width v_j to obtain a rectangle r_j of height $h_j(G)$ and width v_j .

Let $G(R) = (r_j : 1 \leq j \leq q)$; hence (4.3) implies $G(R)$ is similar to G . Let μ be the additional measure needed to form $G(R)$. Since $n_j \geq K$, (4.2) implies

$$(4.4) \quad \mu < h \sum_{j=1}^q v_j \leq hw/K.$$

Let G be a tower and let R be a rectangle. The minimum height of a rectangle in $G' = S^p(G)$ increases without bound as $p \rightarrow \infty$. Thus as p increases, K will increase in the preceding construction to form $G'(R)$. Hence (4.4) implies that p can be chosen large enough to insure that μ is small when forming $G'(R)$.

Let G be an M -tower and let $\varepsilon > 0$. Since $T_t(G)$ is mixing, there exists a positive integer $t(G, \varepsilon)$ such that

$$(4.5) \quad \left| \frac{m(T_t(A) \cap B)}{m(A)m(B)/m(G)} - 1 \right| < \varepsilon, \quad t \geq t(G, \varepsilon)$$

where A and B are ε -rectangles in G .

Let G and g be disjoint similar M -towers, where $g = \gamma G$, $0 < \gamma < 1$. Let $G^* = (G, g)$, and consider the flow $T_t(G^*)$. A rectangle E in G will move up and emerge from the top of G . The definition of the S operator implies that a proportion γ of E will move into G and a proportion $(1 - \gamma)$ will move into g . Since g and G are similar, this proportion of E in g and G will be preserved for all later time t . Also, if F is a rectangle in G , then

$$(4.6) \quad m(T_t(G^*)(E) \cap F) \geq \gamma m(T_t(G)(E) \cap F), \quad t \geq 0.$$

In fact, equality holds in (4.6) for t greater than the height of the highest rectangle in G .

REMARK 4.7. Our construction of an α -mixing flow will make repeated use of the following. Let $\varepsilon > 0$ and $\delta > 0$. Let G_1 be an M -tower and let r be a rectangle disjoint from G_1 . Let $t_1 = t(G_1, \varepsilon)$ and choose $t_2 > t_1$. We can choose a positive integer $p = p(G_1, \varepsilon, t_2)$ so large that if $G_2 = S^p(G_1)$ and T_t extends $T(G_2)_t$, then

$$(4.8) \quad \left| \frac{m(T_t(E) \cap F)}{m(E)m(F)/m(G_1)} - 1 \right| < \varepsilon, \quad t_1 \leq t \leq t_2$$

where E and F are ε -rectangles in G_1 . We can also choose p so large that if μ is the measure needed to form $G_2(r)$, then $\mu < \delta$. Let $G_3 = (G_2, G_2(r))$

and let $\gamma = m(G_2)/m(G_3)$. From (4.6) we obtain

$$(4.9) \quad m(T_t(G_3)(E) \cap F) \geq \gamma m(T_t(G_2)(E) \cap F), \quad t \geq 0,$$

where E and F are ε -rectangles in G_2 . Since an ε -rectangle in G_1 appears as a finite union of ε -rectangles in G_2 , (4.9) also holds for ε -rectangles in G_1 . Since $T_t(G_1) = T_t(G_2)$, (4.9) and the choice of t_1 imply

$$(4.10) \quad m(T_t(G_3)(E) \cap F) > \gamma(1 - \varepsilon)m(E)m(F)/m(G_1), \quad t \geq t_1,$$

where E and F are ε -rectangles in G_1 .

Given $\eta > 0$, we can choose $t_3 \geq \max(t_2, t(G_3, \eta))$ such that

$$(4.11) \quad \left| \frac{m(T_{t_3}(G_3)(E) \cap F)}{m(E)m(F)/m(G_3)} - 1 \right| < \eta, \quad t \geq t_3$$

where E and F are η -rectangles in G_3 .

To construct an α -mixing flow, we begin with an M -tower G_1 whose rectangles have equal widths, a summable sequence $\delta_n \rightarrow 0$, and a sequence $\varepsilon_n \rightarrow 0$ where ε_1 is less than the minimum height of the rectangles of G_1 , and where $\varepsilon_n = \varepsilon_1/2^{n-1}$. The flow is obtained by an iterative construction, at whose k^{th} stage we have an M -tower G_k and $t_{k,0} \geq 0$. (We take $t_{1,0} = 0$.) We divide G_k into disjoint copies $G_{k,1} = \alpha G_k$ and $(1 - \alpha)G_k$. Let $R_{k,1} = V((1 - \alpha)G_k)$, and let $h_{k,1}$ be the height of $R_{k,1}$. Choose $t_{k,1} > \max(t_{k,0}, t(G_k, \varepsilon_k))$, and choose a positive integer a_k so that $a_k h_{k,1} > t_{k,1}$. Let $R_{k,2} = S_{a_k}(R_{k,1})$ and $h_{k,2} = a_k h_{k,1}$. Choose a positive integer b_k so that $1/b_k < \varepsilon_k$, and let $R_{k,3} = S_{b_k}(R_{k,2})$. Thus if T_t extends $T(R_{k,3})_t$ and R is a rectangle in $R_{k,2}$, then

$$(4.12) \quad m(T_{h_{k,2}}(R) \cap R) \geq (1 - 1/b_k)m(R).$$

Since a rectangle in $R_{k,1}$ is a disjoint union of rectangles in $R_{k,2}$, (4.12) also holds for rectangles in $R_{k,1}$.

Now decompose $R_{k,3}$ into b_k equal copies $r_{k,i}$, $1 \leq i \leq b_k$. Choose $t'_{k,1} > h_{k,2}$ and apply Remark 4.7 with $G_1 = G_{k,1}$, $r = r_{k,1}$, $t_1 = t_{k,1}$, $t_2 = t'_{k,1}$, $\varepsilon = \eta = \varepsilon_k$ and $\delta = \delta_k/\varepsilon_k$. Then we get $p_{k,1}$ and $G'_{k,1} = S^{p_{k,1}}(G_{k,1})$ such that if T_t extends $T(G'_{k,1})_t$, then

$$(4.13) \quad \left| \frac{m(T_t(E) \cap F)}{m(E)m(F)/m(G_{k,1})} - 1 \right| < \varepsilon_k,$$

for ε_k -rectangles in $G_{k,1}$ and $t_{k,1} \leq t \leq t'_{k,1}$.

When $k \geq 2$, however, we must choose $p_{k,1}$ a little more carefully. In this case, we have an M -tower $G_{k-1,b_{k-1}+1}$ such that

$$G_k = S^{p_{k-1,b_{k-1}+1}}(G_{k-1,b_{k-1}+1})$$

where $p_{k-1,b_{k-1}+1} = p(G_{k-1,b_{k-1}+1}, \varepsilon_{k-1}, t_{k,0})$, and $t_{k,0} > t(G_{k-1,b_{k-1}+1}, \varepsilon_{k-1})$.

For $k \geq 2$, then, we also choose $p_{k,1} \geq p(G_{k-1,b_{k-1}+1}, \varepsilon_k, t_{k,1}) - p_{k-1,b_{k-1}+1}$.

Thus, since $G_{k,1}$ is an α -size copy of G_k , we see that if T_t extends $T(G'_{k,1})$, then for ε_{k-1} -rectangles E and F in $G_{k-1,b_{k-1}+1}$ and for $t_{k,0} \leq t \leq t_{k,1}$.

$$(4.14) \quad m(T_t(E) \cap F) \geq \alpha(1 - \varepsilon_{k-1})m(E)m(F)/m(G_{k-1,b_{k-1}+1}).$$

Our application of Remark 4.7 also gives us $G''_{k,1} = G'_{k,1}(r_{k,1})$ after adding a set of measure $\mu_{k,1} < \delta_k/b_k$. Let $G_{k,2} = (G'_{k,1}, G''_{k,1})$ and $\gamma_{k,1} = m(G_{k,1})/m(G_{k,2})$. By (4.10) we have that for ε_k -rectangles E and F in $G_{k,1}$, and $t \geq t_{k,1}$,

$$(4.15) \quad m(T_t(G_{k,2})(E) \cap F) \geq \gamma_{k,1}(1 - \varepsilon_k)m(E)m(F)/m(G_{k,1}).$$

Now choose $t_{k,2} > \max(t'_{k,1}, t(G_{k,2}, \varepsilon_k))$ so that for $t \geq t_{k,2}$ and ε_k -rectangles E and F in $G_{k,2}$,

$$(4.16) \quad \left| \frac{m(T_t(E) \cap F)}{m(E)m(F)/m(G_{k,2})} - 1 \right| < \varepsilon_k .$$

Choose $t'_{k,2} > t_{k,2}$ and again apply Remark 4.7 where this time $G_1 = G_{k,2}$, $r = r_{k,2}$, $t_1 = t_{k,2}$, $t_2 = t'_{k,2}$, $\varepsilon = \eta = \varepsilon_k$, and $\delta = \delta_k/b_k$. We get $p_{k,2}$ and $G'_{k,2} = S^{p_{k,2}}(G_{k,2})$ so that conditions corresponding to (4.13) and (4.15) hold. This time, however, we choose $p_{k,2}$ so large that if T_t extends $T(G'_{k,2})_t$, then (4.15) perseveres to give

$$(4.17) \quad m(T_t(E) \cap F) > \gamma_{k,1}(1 - \varepsilon_k)m(E)m(F)/m(G_{k,1}) ,$$

for ε_k -rectangles E and F in $G_{k,1}$, and $t'_{k,1} \leq t \leq t_{k,2}$.

We now continue the construction in this manner until we have G_{k,b_k+1} and $t_{k,b_k} < t'_{k,b_k} < t_{k,b_k+1}$, where $t_{k,b_k} > t(G_{k,b_k}, \varepsilon_k)$ and $t_{k,b_k+1} > t(G_{k,b_k+1}, \varepsilon_k)$. We then choose $t'_{k,b_k+1} > t_{k,b_k+1}$ and p_{k,b_k+1} so that if $G'_{k,b_k+1} = S^{p_{k,b_k+1}}(G_{k,b_k+1})$ and T_t extends $T(G'_{k,b_k+1})$, then for ε_k -rectangles E and F in G_{k,b_k} and for $t'_{k,b_k} \leq t \leq t_{k,b_k+1}$,

$$(4.18) \quad m(T_t(E) \cap F) \geq \gamma_{k,b_k+1}(1 - \varepsilon_k)m(E)m(F)/m(G_{k,b_k})$$

and for ε_k -rectangles E and F in G_{k,b_k+1} and $t_{k,b_k+1} \leq t \leq t'_{k,b_k+1}$,

$$(4.19) \quad \left| \frac{m(T_t(E) \cap F)}{m(E)m(F)/m(G_{k,b_k+1})} - 1 \right| < \varepsilon_k .$$

We then let $G'_{k,b_k+1} = G_{k+1}$ and $t'_{k,b_k+1} = t_{k+1,0}$, and continue with the $(k+1)^{st}$ stage of the construction.

Note that the set added in the k^{th} stage has measure

$$\sum_{i=1}^{b_k} \mu_{k,i} < b_k(\delta_k/b_k) = \delta_k ,$$

so that the total measure we add is finite. Let $(X, \mathcal{B}, m, T)$ denote the flow obtained by iterating the above construction, where the measure has been normalized so that $m(X) = 1$.

We now show that T is α -mixing but not β -mixing for any $\beta > \alpha$. First suppose that A and B are ϵ_k -rectangles in G_k . We show that

$$(4.20) \quad \lim_{t \rightarrow \infty} m(T_t(A) \cap B) \geq \alpha m(A)m(B).$$

For $1 \leq i \leq b_k + 1$, a fraction $\beta \geq \alpha$ of both A and B appears as a disjoint union of ϵ_k -rectangles in $G_{k,i}$. (In fact, $\beta = m(G_{k,i})/m(G_k)$.)

Since T_t extends $T(G'_{k,i})$, we have

$$(4.21) \quad \begin{aligned} m(T_t(A) \cap B) &\geq m(T_t(E) \cap F) \geq (1 - \epsilon_k)m(E)m(F)/m(G_{k,i}) \\ &= \beta(1 - \epsilon_k)m(A)m(B)/m(G_k) \geq \alpha(1 - \epsilon_k)m(A)m(B), \end{aligned}$$

for $t_{k,i} \leq t \leq t'_{k,i}$.

Since T_t extends $T(G'_{k,i})$, we have

$$(4.22) \quad \begin{aligned} m(T_t(A) \cap B) &\geq m(T_t(E) \cap F) \geq \gamma_{k,i-1}(1 - \epsilon_k)m(E)m(F)/m(G_{k,i-1}) \\ &= \beta\gamma_{k,i-1}(1 - \epsilon_k)m(A)m(B) \geq \alpha\gamma_{k,i-1}(1 - \epsilon_k)m(A)m(B). \end{aligned}$$

Now since G_{k+1} is an α -size copy of G_{k+1} , and since αA and αB appear as disjoint unions E and F of ϵ_k -rectangles in $G_{k+1,1}$, condition (4.14) implies that for $t'_{k,b_k+1} \leq t \leq t_{k+1,1}$,

$$(4.23) \quad \begin{aligned} m(T_t(A) \cap B) &\geq m(T_t(E) \cap F) \geq (1 - \epsilon_k)m(E)m(F)/m(G_{k+1,1}) \\ &= \alpha(1 - \epsilon_k)m(A)m(B)/m(G_{k+1}) \geq \alpha(1 - \epsilon_k)m(A)m(B). \end{aligned}$$

For every $\ell > k$, both A and B appear as disjoint unions of ϵ_ℓ -rectangles in G . Then since $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and $\lim_{k \rightarrow \infty} (\min\{\gamma_{k,i}\}) = 1$, conditions (4.21), (4.22) and (4.23) imply (4.20).

Since every set in B can be arbitrarily well approximated by disjoint unions of ε_k -rectangles in G_k for sufficiently large k , (4.20) implies that T is α -mixing.

However, if A and B are disjoint ε_1 -rectangles in G_1 , then ($\forall k > 0$), $A = A_{k,1} \cup A_{k,2}$ where $A_{k,1} \subset G_{k,1}$, $m(A_{k,1}) = \alpha m(A)$, $A_{k,2} \subset R_{k,3}$, $m(A_{k,2}) = (1 - \alpha)m(A)$, and $m(T_{h_{k,2}}(A) \cap B) = \sum_{i=1}^2 m(T_{h_{k,2}}(A_i) \cap B_j)$. By (4.12) $m(T_{h_{k,2}}(A_2) \cap B) < \varepsilon_k m(A_2) = \varepsilon_k (1 - \alpha)m(A)$, and since $h_{k,2}/h_{k,3} = 1/b_k < \varepsilon_k$, $m(T_{h_{k,2}}(A_1) \cap B_2) < \varepsilon_k m(B_2) = \varepsilon_k (1 - \alpha)m(B)$. But since $t_{k,1} \leq h_{k,2} \leq t'_{k,1}$, (4.13) implies that

$$\left| \frac{m(T_{h_{k,2}}(A_1) \cap B_1)}{m(A_1)m(B_1)/m(G_{k,1})} - 1 \right| = \left| \frac{m(T_{h_{k,2}}(A_1) \cap B_1)}{\alpha m(A_1)m(B_1)/m(G_k)} - 1 \right| < \varepsilon_{k_1}.$$

Thus $\lim_{k \rightarrow \infty} m(T_{h_{k,2}}(A) \cap B) = \alpha m(A)m(B) = \lim_{t \rightarrow \infty} m(T_t(A) \cap B)$, so that for all $\beta > \alpha$, T is not β -mixing.

To verify that the flow has zero entropy, we will show that T_1 has zero entropy. It suffices to construct, for each positive integer n , a generator P_n for T_1 with $H(P_n) \leq \frac{1}{n}$. The partition P_n will consist of a sequence of sets $\{A_j^i\}_{i=1}^j$ together with the complement of their union, where A_j^i will be a $\frac{1}{j}$ -rectangle in a suitably chosen $R_{k_j,1}$.

To begin, we choose k_1 so that if A_1^1 is the bottom 1-rectangle of $R_{k_1,1}$, then $m(A_1^1) < \frac{1}{10} m(R_{k_1,1})$, and the entropy of the partition $p_n^1 = (A_1^1, A_1^{1c})$ satisfies $H(p_n^1) < \frac{1}{n}$. We proceed by the following iterative construction. Suppose we have disjoint sets $\{A_j^i\}_{i=1}^j$ and integers $k_1 < k_2 < \dots < k_\ell$ such that

(4.24) for each j , the sets $A_j^1, \dots, A_j^i$ are consecutive $\frac{1}{j}$ -rectangles in $R_{k_j,1}$,

$$(4.25) \quad m\left(\bigcup_{j=1}^{\ell} \bigcup_{i=1}^j A_j^i\right) < \frac{1}{10},$$

and

(4.26) the entropy of the partition $P_n^\ell = \{A_1^1, A_2^1, A_2^2, \dots, A_\ell^\ell\}$ satisfies $H(P_n^\ell) < \frac{1}{n}$. For each j , let $A_j = \bigcup_{i=1}^j A_j^i$. If $k_{\ell+1} > k_\ell$ then each A_j intersects $R_{k_{\ell+1},1}$ in a disjoint union of 1-rectangles and $m((\bigcup_{j=1}^\ell A_j) \cap R_{k_{\ell+1},1}) < \frac{1}{10} m(R_{k_{\ell+1},1})$, since $R_{k_{\ell+1},1}$ is obtained from a copy of $G_{k_{\ell+1}}$. Therefore, we may choose a 1-rectangle $A_{\ell+1}$ in $R_{k_{\ell+1},1}$ disjoint from $\bigcup_{j=1}^\ell A_j$ and subdivide it into consecutive $\frac{1}{\ell+1}$ -rectangles $A_{\ell+1}^1, \dots, A_{\ell+1}^{\ell+1}$. If $k_{\ell+1}$ is chosen sufficiently large, then conditions (4.25) and (4.26) can be made to hold with $\ell + 1$ replacing ℓ . Letting P_n denote the partition obtained by iterating this construction, we see that $H(P_n) \leq \frac{1}{n}$.

Now let E be an ε -rectangle in G_k , and fix $\delta > 0$. We construct a set $F \subset \bigvee_{i=-\infty}^{\infty} T_1^{-i} P_n$ such that $m(E \Delta F) < \delta \cdot m(E)$. Choose j so that $k_j > k$ and $\frac{4}{j} < \delta \cdot \varepsilon$. Then a set $E_1 \subset E$ with $m(E_1) = (1 - \alpha)m(E)$ appears in $R_{k_j,1}$ as a disjoint union of ε -rectangles and can therefore be approximated by T_1 -translates of the sets $A_j^1, \dots, A_j^j$ to within $\frac{\delta}{2} \cdot m(E_1)$. Similarly, a set $E_2 \subset E \setminus E_1$ with $m(E_2) = (1 - \alpha)m(E \setminus E_1)$ appears in $R_{k_{j+1},1}$ as a union of ε -rectangles and so can be approximated to within $\frac{\delta}{2} \cdot m(E_2)$ by T_1 -translates of the sets $A_{j+1}^1, \dots, A_{j+1}^{j+1}$. Continuing in this manner m times, where $\alpha^m < \frac{\delta}{2}$, we obtain the desired approximation of E . This then implies that P_n is a generator for T_1 .

5. Eventually Independent Sequences

Let T be a weak mixing transformation. An increasing sequence of positive integers $\sigma = \{c_i\}_{i=1}^\infty$ is an *eventually independent sequence* (e.i.s.) for T if for each finite partition P and $\varepsilon > 0$ there exists a finite partition Q with the same number of sets as P and a positive integer N such that $|P - Q| < \varepsilon$ and $(T^{c_i} Q : i > N)$ is an independent sequence of partitions. It is

shown in [2] that every weak mixing transformation admits an e.i.s. An e.i.s. is an isomorphism invariant.

We now fix $\alpha \in (0,1)$ and modify the construction in section 4 to show that there exist uncountably many pair-wise non-isomorphic α -mixing flows. Suppose $\{S^{(n)}\}_{n=1}^{\infty}$ is a sequence of α -mixing flows. Then each $S_1^{(n)}$ is α -mixing and therefore weak mixing and so admits an e.i.s. For each n , fix an e.i.s. $\sigma_n = \{c_i^{(n)}\}_{i=1}^{\infty}$ for $S_1^{(n)}$. We will construct an α -mixing flow T such that for all n , σ_n is not an e.i.s. for T_1 and hence, $S^{(n)}$ and T are not isomorphic. Since the sequence $\{S^{(n)}\}_{n=1}^{\infty}$ is arbitrary, we will have the desired result.

We construct the flow T as in section 4 in such a way that the heights $h_{k,2}$ of the rectangles $R_{k,2}$ form an increasing sequence of integers such that $h_{1,2} \in \sigma_1$, and if $h_{k-1,2} \in \sigma_j$, then $h_{k,2} \in \sigma_1$ if $h_{k-1,2}$ is the first integer in this sequence chosen from σ_j , and $h_{k,2} \in \sigma_{j+1}$ otherwise. This is accomplished as follows. At the k^{th} stage of the construction of T , let σ_ℓ be the e.i.s. from which we want to choose $h_{k,2}$. Choose a_k so that $w_k h_{k,1}/a_k < \frac{1}{2^k}$ and so that there is a $c_j^{(\ell)} \in \sigma_\ell$ satisfying $0 < c_j^{(\ell)} - a_k h_{k,1} < h_{k,1}$. Then we can add a rectangle of height less than $h_{k,1}$ to $R_{k,2}$ to make $R_{k,2}$ have height $c_j^{(\ell)}$, and the total measure added in this way in the construction of T is finite. Now let A be a rectangle in G_1 with $m(A) < (1 - \alpha)/2$. Then if the b_k are chosen sufficiently large, we have, for each k ,

$$m(T_{h_{k,2}}(A) \cap A) > \frac{2}{3} m(A \cap R_{k,3}) > \frac{2}{3} m(A)(1 - \alpha) > \frac{4}{3} m(A)^2.$$

Consequently, no subsequence of $\{h_{k,2}\}_{k=1}^{\infty}$ is an e.i.s. for T_1 . Since, for every n , $\sigma_n \cap \{h_{k,2}\}_{k=1}^{\infty}$ is an infinite sequence, we see that σ_n is not an e.i.s. for T_1 .

References

- [1] FRIEDMAN, N.A., *Bernoulli shifts induce Bernoulli shifts*, Advances in Math., 10, 1973, 39-48.
- [2] FRIEDMAN, N.A., *Eventually independent sequences*, Israel J. Math., 33, 1979, 310-316.
- [3] FRIEDMAN, N.A., and ORNSTEIN, D.S., *On partial mixing transformations*, Indiana Univ. Math. J., 20, 1971, 767-775.
- [4] FRIEDMAN, N.A., and ORNSTEIN, D.S., *On mixing and partial mixing*, Illinois J. Math., 16, 1972, 61-68.
- [5] ORNSTEIN, D.S., *Imbedding Bernoulli shifts in flows*, Springer-Verlag Lecture Notes Math. 160, 1970, 178-218.
- [6] SHIELDS, P., *Cutting and independent stacking of intervals*, Math. Systems Theory 7, 1973, 1-4.

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