

ALMOST-PERIODICITY FOR OPERATOR-VALUED FUNCTIONS¹

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Résumé

Dans cette note, on étudie des fonctions presque périodiques $A(t)$ définies sur la droite réelle, à valeurs dans l'algèbre des opérateurs linéaires continus d'un espace de Banach et on établit une liaison entre la presque périodicité "forte" et celle "uniforme". Le résultat est semblable à celui reliant la presque périodicité faible avec celle forte pour les fonctions à valeurs dans un espace de Banach arbitraire.

Introduction

In this note, we continue previous work [6]; our main concern here is to establish a connection between strong and uniform almost-periodicity for operator-valued functions defined on the real line. There is a striking analogy with Amerio's work on weakly almost-periodic functions [1].

1. Let be E a Banach space and $[E]$ be the Banach algebra of linear continuous mappings of E into itself ([4], [5]). Consider functions $A(t)$, $\mathbb{R} \rightarrow [E]$; we study (Bochner) almost-periodicity of $A(t)$ in strong and uniform topology. Remark that strongly almost-periodic $A(t)$ is uniformly bounded on $\mathbb{R}$. Also, uniform on $\mathbb{R}$ limit of almost-periodic functions $A_n(t)$ in the strong sense,

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is likewise almost-periodic in the strong sense. Furthermore, if $A(t)$, $R \rightarrow [E]$ is strongly a.p. and if $(s_n)_1^\infty \subset R$ is a real sequence such that:

$\lim_{n \rightarrow \infty} A(t+s_n)e = B(t)e$ holds, $\forall e \in E$, pointwise on R , then the convergence is actually uniform on R and $B(t)$ is again strongly a.p. Moreover, it is easy to

find that: $\sup_{t \in R} \|A(t)\|_{[E]} = \sup_{t \in R} \|B(t)\|_{[E]}$. Let us prove now the following (preliminary).

PROPOSITION 1. Let $A(t)$, $R \rightarrow [E]$ be uniformly continuous on R in the strong sense and the range $\{A(t), t \in R\}$ in $[E]$ be relatively compact. Then $A(t)$, $R \rightarrow [E]$ is uniformly continuous on R in the uniform sense.

In fact if the result were false, $\exists \sigma > 0$ such that $\forall n = 1, 2, \dots$, $\exists t_1^n, t_2^n \in R$, $|t_1^n - t_2^n| < \frac{1}{n}$ and $\|A(t_1^n) - A(t_2^n)\|_{[E]} > \sigma$. Using relative compactness we find a sequence $n_1 < n_2 < \dots$ such that $\lim_{p \rightarrow \infty} A(t_1^{n_p}) = A_1$ exists in $[E]$ -sense. Next, there is a sequence $p_1 < p_2 < \dots$, such that $\lim_{\ell \rightarrow \infty} A(t_2^{n_p \ell}) = A_2$ exists again in $[E]$. It follows, with notations $t_1^{n_p \ell} = \tau_1^\ell$, $t_2^{n_p \ell} = \tau_2^\ell$ that: $\lim_{\ell \rightarrow \infty} A(\tau_1^\ell) = A_1$, $\lim_{\ell \rightarrow \infty} A(\tau_2^\ell) = A_2$ in $[E]$ -sense. Also $\|A_1 - A_2\|_{[E]} > \sigma$ (as easily seen). Hence, for at least one element $e \in E$, we have $A_1 e \neq A_2 e$. Consider however the sequence $A(\tau_1^\ell)e - A(\tau_2^\ell)e$; remark that $|\tau_1^\ell - \tau_2^\ell| \rightarrow 0$ as $\ell \rightarrow \infty$. From uniform continuity on R of $A(\tau)e$, $R \rightarrow E$, we deduce that $\lim_{\ell \rightarrow \infty} \|A(\tau_1^\ell)e - A(\tau_2^\ell)e\|_E = 0$. But $A(\tau_1^\ell)e \rightarrow A_1 e$ and $A(\tau_2^\ell)e \rightarrow A_2 e$ in E implies that $\|A_1 e - A_2 e\|_E = 0$, a contradiction.

Next we remark the following simple:

PROPOSITION 2. If $A(t)$, $B(t)$ are both strongly almost-periodic functions, $R \rightarrow [E]$, then their product $(A \cdot B)(t) = A(t) \cdot B(t)$ is also strongly a.p., from R to $[E]$.

In fact, for any $e \in E$, it is: $(A \cdot B)(t)e = A(t)(B(t)e)$. Now, $B(t)e$ is a.p., from R to E , so that applying [6] we get the result.

We end this section with following:

PROPOSITION 3. Let be $A(t)$ a strongly a.p. function, $R \rightarrow [E]$ and M be a relatively compact set in E . Then, the family of a.p. functions, $R \rightarrow E$, defined by $\{A(t)e, e \in M\}$ is a relatively compact family with respect to the uniform convergence on R .

In fact, given any sequence $(e_n)_1^\infty \subset M$, there exists a subsequence $(e_{n_j})_1^\infty$, such that $e_{n_j} \rightarrow e \in E$. Therefore, the function-sequence $(A(t)e_{n_j})_1^\infty$ converges to $A(t)e$, uniformly on R , because obviously $A(t)$ is uniformly bounded, $R \rightarrow [E]$.

2. In this section we state and prove the main result of this paper, namely the following:

THEOREM. Let $A(t)$, $R \rightarrow [E]$, be strongly almost-periodic and the range $\{A(t), t \in R\}$ be relatively compact in $[E]$. Then $A(t)$ is almost-periodic from R to $[E]$ (i.e. uniformly).

THE PROOF. We show that for any real sequence $\{\alpha'_n\}_1^\infty$ there is a subsequence $\{\alpha_n\}_1^\infty$ in such a way that $\lim_{n \rightarrow \infty} A(t + \alpha_n) = B(t)$ will exist in $[E]$, uniformly on R . Let us denote by $\{\eta_k\}_1^\infty$ the sequence of all rational numbers. Using relative compactness of the range we may find a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ in such a way that:

$$(*) \quad \lim_{n \rightarrow \infty} A(\alpha_n + \eta_k) = B_k$$

will exist in $[E]$, for all $k = 1, 2, \dots$

Before continuing the proof, let us give the general:

LEMMA 1. Let be $x(t)$ an almost-periodic function from R into a Banach space X . Let also $(s_n)_1^\infty$ be a real sequence such that $\lim_{n \rightarrow \infty} x(s_n + \eta_k)$ exists for all $k = 1, 2, \dots$. Then the function sequence $\{x(s_n + t)\}_{n=1}^\infty$ is uniformly convergent on R (in strong X -sense).

In fact, otherwise, $\exists \varepsilon_0 > 0$ such that for some increasing sequences of natural numbers $m_1 < m_2 < \dots$ and $n_1 < n_2 < \dots$ and for some real numbers $t_1, t_2, \dots$ the lower bound $\|x(s_{n_N} + t_N) - x(s_{m_N} + t_N)\|_X > \varepsilon_0$, $N = 1, 2, \dots$, holds. Using almost-periodicity of $x(t)$ we may assume, after some extraction of subsequences, that $\lim_{N \rightarrow \infty} x(s_{n_N} + t) = y_1(t)$, $\lim_{N \rightarrow \infty} x(s_{m_N} + t) = y_2(t)$, uniformly on $\mathbb{R}$, in X -sense. Obviously also it is:

$$\begin{aligned} \|y_1(t_N) - y_2(t_N)\| &\geq \|x(s_{n_N} + t_N) - x(s_{m_N} + t_N)\| - \|[y_1(t_N) - x(s_{n_N} + t_N) \\ &\quad + x(s_{m_N} + t_N) - y_2(t_N)]\| \\ &\geq \varepsilon_0 - \frac{2\varepsilon_0}{3} = \frac{\varepsilon_0}{3} \end{aligned}$$

for large N . On the other hand

$$\lim_{N \rightarrow \infty} x(s_{n_N} + \eta_k) = \lim_{N \rightarrow \infty} x(s_{m_N} + \eta_k), \quad \forall k = 1, 2, \dots$$

as both (s_{n_N}) , (s_{m_N}) are subsequences of (s_n) . Therefore $y_1(\eta_k) = y_2(\eta_k)$, $k = 1, 2, \dots$ and by their obvious continuity on $\mathbb{R}$, $y_1(t) = y_2(t)$, $\forall t \in \mathbb{R}$, a contradiction.

We now proceed to the following:

LEMMA 2. The limit in (*) $A(\alpha_n + \eta_k) \rightarrow B_k$ in $[E]$ -sense — is uniform with respect to $k = 1, 2, \dots$

If not, $\exists \sigma > 0$ such that for some subsequences $k_1 < k_2 < \dots$, $\ell_1 < \ell_2 < \dots$ it is $\|A(\alpha_{k_n} + \eta_{j_n}) - A(\alpha_{\ell_n} + \eta_{j_n})\|_{[E]} \geq \sigma$ for some $j_n \in \mathbb{N}$, $\forall n = 1, 2, \dots$. Let be $\alpha_{k_n} + \eta_{j_n} = \beta_n$, $\alpha_{\ell_n} + \eta_{j_n} = \gamma_n$, $n = 1, 2, \dots$. It follows that: $\|A(\beta_n) - A(\gamma_n)\|_{[E]} \geq \sigma$, $n = 1, 2, \dots$

Using relative compactness of the range of $A(t)$ in $[E]$ we find:

$\exists (\beta_{n_p}) \subset (\beta_n)$ such that $A(\beta_{n_p}) \rightarrow A_1$ in $[E]$. Next, $\exists (\gamma_{n_{p_j}}) \subset (\gamma_{n_p}) \subset (\gamma_n)$ such that $A(\gamma_{n_{p_j}}) \rightarrow A_2$ in $[E]$. If denote: $\beta_{n_{p_j}} = \beta_j^1$, $\gamma_{n_{p_j}} = \gamma_j^1$, $j \in \mathbb{N}$,

we get $\|A(\beta_j^1) - A(\gamma_j^1)\|_{[E]} \geq \sigma$, $j = 1, 2, \dots$ and also $A(\beta_j^1) \rightarrow A_1$, $A(\gamma_j^1) \rightarrow A_2$ in $[E]$ -sense. Therefore $\|A_1 - A_2\|_{[E]} \geq \sigma$ and $\exists \bar{x} \in E$ such that $A_1 \bar{x} \neq A_2 \bar{x}$. Consider now the almost-periodic function $A(t)\bar{x}$, $R \rightarrow E$. Using (*) above, we see that $A(\alpha_n + \eta_k)\bar{x} \rightarrow B_k \bar{x}$ as $n \rightarrow \infty$, in E -sense, $\forall k = 1, 2, \dots$. Using Lemma 1 we find that the function-sequence $\{A(\alpha_n + t)\bar{x}\}_{n=1}^\infty$ is E -convergent uniformly on R .

Consider now the following sequences of functions: $\{A(t + \beta_j^1)\bar{x}\}_{j=1}^\infty$, $\{A(t + \gamma_j^1)\bar{x}\}_{j=1}^\infty$. Using almost-periodicity of $A(t)\bar{x}$, after two more extractions of subsequences we arrive at sequences $(\beta_j^!)_1^\infty$, $(\gamma_j^!)_1^\infty$ of real numbers, such that $(\beta_j^!) \subset (\beta_j^1)$, $(\gamma_j^!) \subset (\gamma_j^1)$, and $\lim_{j \rightarrow \infty} A(t + \beta_j^!)\bar{x} = \bar{x}_1(t)$, $\lim_{j \rightarrow \infty} A(t + \gamma_j^!)\bar{x} = \bar{x}_2(t)$, uniformly on R .

Remark now that $\beta_j^! = \alpha_j^! + \eta_j^!$, $\gamma_j^! = \alpha_j'' + \eta_j^!$ where $(\alpha_j^!)_1^\infty$, $(\alpha_j'')_1^\infty$, are subsequences of $(\alpha_k)_1^\infty$. We have estimate:

$$\begin{aligned} \|A(t + \beta_j^!)\bar{x} - A(t + \gamma_j^!)\bar{x}\|_E &\leq \|A(t + \eta_j^! + \alpha_j^!)\bar{x} - A(t + \eta_j^! + \alpha_{k_j}^!)\bar{x}\|_E \\ &\quad + \|A(t + \eta_j^! + \alpha_{k_j}^!)\bar{x} - A(t + \eta_j^! + \alpha_j'')\bar{x}\|_E. \end{aligned}$$

Actually, due to the uniform convergence on R of the function-sequence $\{A(\alpha_n + t)\bar{x}\}$ we see that for large j it is $\|A(t + \beta_j^!)\bar{x} - A(t + \gamma_j^!)\bar{x}\|_E < \varepsilon$ (for arbitrarily given $\varepsilon > 0$). It follows that $\bar{x}_1(t) = \bar{x}_2(t)$, $\forall t \in R$, for $t = 0$ also, $A_1 \bar{x} = A_2 \bar{x}$, a contradiction. We may now proceed to the

END OF PROOF OF THE THEOREM. We shall see that the function-sequence $\{A(t + \alpha_n)\}_1^\infty$ is a Cauchy-sequence in $[E]$, uniformly on R . Let us write in fact the relation

$$\begin{aligned} A(t + \alpha_n) - A(t + \alpha_m) &= [A(t + \alpha_n) - A(\eta_k + \alpha_n)] + [A(\eta_k + \alpha_n) - A(\eta_k + \alpha_m)] \\ &\quad + [A(\eta_k + \alpha_m) - A(t + \alpha_m)] \end{aligned}$$

where the rationals η_k will be choosen near to t .

Precisely: let be given $\varepsilon > 0$; consider $\delta(\varepsilon/3)$ given by Proposition 1. For a given $t \in \mathbb{R}$ take a rational η_k such that $|\eta_k - t| < \delta$. It results that

$$\|A(t+\alpha_n) - A(\eta_k+\alpha_n)\|_{[E]} < \frac{\varepsilon}{3}, \quad \|A(t+\alpha_m) - A(\eta_k+\alpha_m)\|_{[E]} < \frac{\varepsilon}{3}, \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{N}.$$

Hence, we obtain

$$\|A(t+\alpha_n) - A(t+\alpha_m)\|_{[E]} < 2 \frac{\varepsilon}{3} + \|A(\eta_k+\alpha_n) - A(\eta_k+\alpha_m)\|_{[E]}.$$

Here we apply also Lemma 2 and find that

$$\|A(\eta_k+\alpha_n) - A(\eta_k+\alpha_m)\|_{[E]} < \frac{\varepsilon}{3}$$

for large m and n . Therefore

$$\|A(t+\alpha_n) - A(t+\alpha_m)\|_{[E]} < \varepsilon$$

for $n, m > n_\varepsilon$, $\forall t \in \mathbb{R}$, which proves the theorem.

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