

RESEARCH PROBLEMS IN FUNCTION THEORY
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Résumé

On trouvera ici un recueil de problèmes ouverts en théorie des fonctions proposés à l'occasion de la vingtième session du Séminaire de mathématiques supérieures tenue à l'Université de Montréal durant l'été 1981.

The problems presented here were proposed by some of the participants of the Séminaire de mathématiques supérieures of the Université de Montréal held from August 3 to 21, 1981. On behalf of all the participants, we express our appreciation to the Gouvernement du Québec and to N.A.T.O. for their generous support.

1. Let $\alpha \in (0, \pi/2)$ be fixed. Then there exists at most one natural number $n \geq 2$ such that

$$n \operatorname{tg} \alpha = \operatorname{tg} n\alpha .$$

REMARK. The conjecture is equivalent to the following statement: given any $x_0 \in (0, 1)$, there is at most one integer $n \geq 1$ such that

$$U'_n(x_0) = 0$$

where $U_n(x)$ is the n^{th} Tchebychev polynomial of the second kind.

(V. Kasten, G. Schmieder)

2. Let R be an open Riemann surface (not planar) and let H_{LS} be the class of locally univalent holomorphic functions on R . For $f, g \in H_{LS}$, let us say that $f < g$ iff for all $U \subset R$, $f|_U$ is univalent implies that $g|_U$ is univalent. Using Zorn's lemma, it is possible to prove the existence of maximal elements in H_{LS} . The question is to exhibit such a maximal element when $R = T \setminus \{p\}$ where T is a torus and $p \in T$.

(G. Schmieder)

3. Let $V_n = \{(a_2, \dots, a_n) \in \mathbb{C}^{n-1} : \text{there exists } z + a_2 z^2 + \dots + a_n z^n + \dots \in S\}$, $U = \{w : |w| < 1\}$, $A = \{w : \frac{1}{2} < |w| < 1\}$ and $a_2(w), \dots, a_n(w)$ be analytic in U . Is it true that $(a_2(w), \dots, a_n(w)) \in \overset{\circ}{V}_n$ for all $w \in A$ implies that $(a_2(w), \dots, a_n(w)) \in \overset{\circ}{V}_n$ for all $w \in U$?

REMARK. The statement is true if S is replaced by the schlicht polynomials of degree n .

(V. Kasten)

4. Let $\sum_{k=0}^{\infty} a_k z^k$ be a formal power series and set $S_n(z) = \sum_{k=0}^n a_k z^k$. Prove or disprove the following statement. If there exists $\varepsilon > 0$ with the following property: for any n , there exists a sector of opening $n^{\varepsilon - \frac{1}{2}}$ containing no zero of S_n , then $\sum_{k=0}^{\infty} a_k z^k$ defines an entire function. If the statement is true, must the entire function be of order zero?

(T. Ganelius)

5. Let $f(z)$ be analytic in $|z| < 1$ and continuous in $|z| \leq 1$, let P_{n-1} be the polynomial of degree $n-1$ interpolating f at the n^{th} roots of unity and let S_{n-1} be the n^{th} partial sum of its Taylor series at the origin. Assume that

$$\lim_{n \rightarrow \infty} (P_{n-1}(z) - S_{n-1}(z)) = 0$$

uniformly in any closed subdomain of $|z| < \rho^2$. Does f have an analytic continuation to $|z| < \rho$?

(J. Szabados)

6. For $f = \sum_{n=1}^{\infty} a_n z^n$ and $g = \sum_{n=1}^{\infty} b_n z^n$, let $f * g = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n$ and let K_R denote the class of close to convex functions with real coefficients. Is it true that

$$K_R * K_R \subseteq K_R ?$$

REMARK. It is known that if K denote the close to convex functions and S^* the starlike ones, then $S^* * S^* \subseteq S^*$, $K * S^* \subset K$ and $K * K \not\subseteq K$.

(D. Bshouty)

7. Let f be holomorphic in an open set D containing the compact set E and denoted by $r_n(f)$ its distance to the rational functions of degree at most n (in the supremum over E norm). Is it true that

$$\liminf_n (r_n(f))^{1/n} \leq e^{-2/C(E,F)}$$

where F is the complement of D and $C(E,F)$ is the capacity of the condenser (E,F) ? Is it true that

$$\sup_f \liminf_n (r_n(f))^{1/n} = e^{-2/C(E,F)}$$

as f varies in the above class? The special case where E is the closed unit disk would be of interest.

REMARK. It is known that

$$\sup_f \limsup_n (r_n(f))^{1/n} = e^{-1/C(E,F)}.$$

(A.A. Gončar)

8. Let f be holomorphic in an open set containing the compact set E and admit analytic continuation to $\overline{\mathbb{C}} \setminus \{z_1, z_2, \dots, z_p\}$. Let

$$S_f = \{F \subset \mathbb{C} : F \text{ is closed and } f \text{ is single-valued in } \overline{\mathbb{C}} \setminus F\}$$

and

$$q_f = \inf_{F \in S_f} e^{-1/C(E,F)}.$$

Is it true that $\lim_n (r_n(f))^{1/n}$ exists and that its value is q_f^2 ?

REMARK. The result is true if E is an interval and if $z_1, z_2, \dots, z_p$ are real. Also known is the fact that, for any $F \in S_f$,

$$\limsup_n (r_n(f))^{1/n} \leq e^{-1/C(E,F)}.$$

(A.A. Gončar)

9. Let R_n denote the rational functions of degree at most n and $H_p^*(U)$ denote the class of functions f holomorphic in the unit disk U and such that $g(z) = (1-z^2)^{-1}f(z)$ belongs to the Hardy class $H_p(U)$ ($p > 1$). Prove that

$$\left(\sup_{\substack{f \in H_p^*(U) \\ \|g\|_p = 1}} \inf_{r_n \in R_n} \sup_{-1 < x < 1} |f(x) - r_n(x)| \right)^{\frac{1}{2}} = e^{-\pi}$$

where $1/p + 1/q = 1$.

(F. Stenger)

10. Given natural numbers $\{n_k\}_{k=1}^N$, let P_N be a projection from $C[a, b]$ to the linear span of $\{1, t^{n_k}\}_{k=1}^N$. Is it true that

$$\lim_{N \rightarrow \infty} \|P_N\| = \infty?$$

(The norms are the supremum on $[a, b]$ norms.) Is it true that

$$\|p_N\| \geq C \log N$$

where $C > 0$ is some constant?

REMARK. The result is known if $n_k = k$ or in the periodic case.

(B. Shekhtman)

11. Let $f(z) = z + a_2 z^2 + \dots \in S$ be such that f^{-1} admits analytic continuation to a function belonging to S . Is it true that f^{-1} is starlike when f is? Is it true that f^{-1} is convex when f is?

(T.S. Taha)

12. Let G be a domain bounded by a rectifiable Jordan curve Γ and, for $p \geq 1$, let $E_p(G)$ denote the class of functions holomorphic in G with boundary values $f(z)$ in $L^p(\Gamma)$. Characterize domains G such that for any $p \geq 1$ there exists a non constant function $f \in E_p(G)$ with $f(z) \geq 0$ a.e. on Γ .

(D. Khavinson)

13. Let π_n denote the polynomials of degree n at most and $r_n(f) = \inf\{\|f - p_n/q_n\| : p_n, q_n \in \pi_n\}$, $E_n(f) = \inf\{\|f - t_n\| : t_n \in \pi_n\}$ where $\|\cdot\|$ denotes the supremum on the closed unit disk. Given a sequence δ_n of positive numbers decreasing to zero, does there exist an entire function f such that

$$r_n(f) < \delta_n E_{2n}(f)$$

for all sufficiently large n ?

(P. Borwein)

14. Let $p > 1$ and $H_p(U)$ denote the Hardy class in the unit disk U . Consider

$$\sigma_{n,p} = \inf_{\substack{w_j \in \mathbb{C} \\ x_j \in U}} \sup_{\substack{f \in H_p(U) \\ \|f\|_p=1}} \left| \int_{-1}^1 f(x) dx - \sum_{j=1}^{2n+1} w_j f(x_j) \right|$$

$$\delta_{n,p} = \inf_{\rho > 0} \sup_{\substack{f \in H_p(U) \\ \|f\|_p=1}} \left| \int_{-1}^1 f(x) dx + 2 \log \rho \sum_{j=-n}^n \frac{\rho^j}{(1+\rho^j)^2} f\left(\frac{1-\rho^j}{1+\rho^j}\right) \right|.$$

Is it true that

$$\lim_{n \rightarrow \infty} \left(\frac{\sigma_{n,p}}{\delta_{n,p}} \right)^{\frac{1}{\sqrt{n}}} = 1 ?$$

(F. Stenger)

15. Given $f, g \in C[0,1]$, let $f * g(x) = \int_0^x f(t)g(x-t)dt$. For a fixed f , in order that the sequence of functions $f, f * f, f * f * f, \dots$ span $C[0,1]$, it is necessary that $f(0) \neq 0$. Is it also sufficient?

(D.J. Newman)

16. Given four non overlapping unit squares in a plane, prove that the diameter of their union is minimum when they are placed so as to form a square.

(D.J. Newman)

17. Let $D_0 = \{z : |z| < 1\}$ and $\{D_i\}_{i \geq 1}$ be a sequence of disjoint open disks such that $\overline{D_0} = \overline{\bigcup_i D_i}$ and $\sum_i \text{diam } D_i < \infty$. Then the measure $d\mu = dz|_{\partial D_0} - \sum_i dz|_{\partial D_i}$ annihilates all rational functions on $X = \overline{D_0} \setminus \bigcup_{i=1}^{\infty} D_i$. If $f \in C(X)$ and $\int_X r(z)f(z)d\mu(z) = 0$ for all rational functions r , is it true that f is in the uniform closure of the rationals?

REMARK. The answer is yes if $f \in \text{Lip } 1$.

(D. Luecking)

18. Let μ be a positive measure with compact support $K \subset \mathbb{R}$. Prove that there is a unique domain $D \subset \mathbb{C}$ such that $K \subset D$ and that

$$\int_K f d\mu = \iint_D f(x+iy) dx dy$$

for all functions f holomorphic in D .

(F. Stenger)

19. Let P_n be the class of polynomial p_n of degree at most n such that $p_n(1) = 0$ and that $\|p_n\| = \max_{|z|=1} |p_n(z)| = 1$. It is known that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that $n - c_1/n \leq \sup_{p_n \in P_n} \|p'_n\| \leq n - c_2/n$. Obtain the value of c such that

$$\sup_{p_n \in P_n} \|p'_n\| = 1 - c/n + o(1/n).$$

(A. Giroux, Q.I. Rahman)

20. Let $\prod_{k=1}^n (z - c_k) / (1 - \bar{c}_k z) = \sum_{k=0}^{\infty} a_k z^k$ where $|c_k| \leq 1$ for $k = 1, 2, \dots, n$.

Is it true that

$$\max_{k \geq 0} |a_k| = \max_{0 \leq k \leq 2n-1} |a_k| ?$$

(W. Fuchs)

21. A quadrilateral Q is a Jordan domain with four prescribed corners. If it is mapped onto a rectangle with sides a and b , its modulus b/a is a conformal invariant. Suppose that a mapping f is such that modulus of $Q =$ modulus of $f(Q)$ for any square Q . Is it true that f is necessarily conformal?

REMARK. The answer is yes if "square" is replaced by "rectangle" but no if "square" is replaced by "square" is replaced by "square with sides parallel to the axis".

(F.W. Gehring)

22. The convolution $f * g = z + \sum_{n=2}^{\infty} a_n b_n z^n$ of a function $f = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and $g = z + \sum_{n=2}^{\infty} b_n z^n \in C$ (convex) need not be univalent as the example $\bar{f} = z(1-iz)^{i-1}$, $\bar{g} = \log \frac{1}{1-z}$ shows (J. Krzyz, Z. Lewandowski, Bull. Acad. Polon. Sci. 11 (1962), 447-448). However $f * \bar{g} \in S$ provided that $|f(z)| < 1e^{\pi/2}(1+e^{-\pi/2})^2$. Is it true that there exists a constant $M > 1$ such that $|f(z)| < M$ and $f \in S$ implies that $f * g \in S$ for any $g \in C$?

(J. Waniurski)

23. For a polynomial of degree n, P_n , is it true that

$$\int_{|z|<1} \frac{|P'_n|}{1+|P_n|^2} |dz|^2 \leq An^{1/2-\epsilon}$$

for some positive number ϵ ?

REMARK. This conjecture goes back to Littlewood (J. Lond. Math. Soc., 1952) who proved that

$$\int_{|z|<1} \frac{|P'_n|}{1+|P_n|^2} |dz|^2 \leq \pi n^{\frac{1}{2}}.$$

(W.K. Hayman)

24. Let $f \in S$ be such that its inverse function f^{-1} admits analytic continuation to a function also in S . If $f = z + \sum_{n=2}^{\infty} a_n z^n$, is it true that $|a_2| \leq \sqrt{2}$? Also, what is $\limsup_n \frac{|a_n|}{n}$? And what is the Koebe constant $\inf\{|w| : f(z) \neq w\}$ for this class S^2 ?

(D.A. Brannan)

25. Let f be meromorphic with simple poles and such that $N(r, 0, f') = o(T(r))$. Denote its order by ρ . Is it true that 2ρ is an integer and that the plane can be divided in sectors of opening π/ρ in each of which f' grows exponentially in annuli $r < |z| < 2r$?

(D. Drasin)

26. The following result is conjectured. Let $p(z)$, $q(z)$ be two non constant polynomials of the same degree. If

$$p(z)(p(z)-1) = 0 \iff q(z)(q(z)-1) = 0 ,$$

(without counting multiplicities) then either $p \equiv q$ or $p + q \equiv 1$.

(C.C. Yang)

27. Let $a > 0$, $b > 0$ and $\alpha > 0$, $\beta > 0$ with $\alpha + \beta = 1$. Define

$$A_{\alpha}(a,b) = \alpha a + \beta b , \quad G_{\alpha}(a,b) = a^{\alpha} b^{\beta}$$

and denote

$$\begin{aligned} G_{\alpha}(a,b) &= a_1 , & A_{\alpha}(a,b) &= b_1 , \\ G_{\alpha}(a_i,b_i) &= a_{i+1} , & A_{\alpha}(a_i,b_i) &= b_{i+1} , & i &= 1,2,\dots \end{aligned}$$

Then it is easy to show that $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} b_i = M(a,b)$. It is conjectured that:

$$M_{\alpha}(a,b) = \frac{\pi}{\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{\alpha}(x^2+b^2)^{\beta}}} .$$

REMARK. Gauss proved this when $\alpha = \frac{1}{2}$, and $M = M_{\frac{1}{2}}(a,b)$ is the so called arithmetic-geometric mean of a and b .

(G.D. Song)

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