

A PARTIAL DIFFERENTIAL EQUATION WITH ALMOST-PERIODIC RIGHT-HAND SIDE

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RÉSUMÉ

Considérons l'équation

$$u_t(x,t) + \sum_{\ell,k=1}^n a_{\ell k} u_{x_\ell x_k} + \sum_{\ell=1}^n a_\ell u_{x_\ell} = f(x,t)$$

où $x = (x_1 \dots x_n) \in \mathbb{R}^n$, $-\infty < t < +\infty$; a_ℓ sont des coefficients constants réels;
 $a_{\ell k}$ sont des coefficients constants complexes tels que:

$$\operatorname{Re} \sum_{\ell,k=1}^n a_{\ell k} s_\ell s_k \geq c_0 \sum_{\ell=1}^n s_\ell^2 \quad \text{où } c_0 > 0, \quad s = (s_1, \dots, s_n) \in \mathbb{R}^n; \quad f(x,t),$$

$-\infty < t < +\infty$, est une fonction presque périodique de $L^2(\mathbb{R}^n)$ au sens de Bochner.

Pour une classe de solutions faibles continues $u(x,t)$, $-\infty < t < +\infty \rightarrow L^2(\mathbb{R}^n)$,
on démontre que toute solution bornée de $L^2(\mathbb{R}^n)$ est aussi presque périodique
dans $L^2(\mathbb{R}^n)$ en utilisant la méthode des projections orthogonales "concrètes"
comme dans un travail précédent de l'auteur (voir par exemple: Rend. Acc. Naz.
Lincei (8), 32 (1962), pp. 30-37).

INTRODUCTION

Let $u'(t) = Au(t) + f(t)$ be a non-homogenous differential equation in the
euclidian n -dimensional space $\mathbb{R}^n$, where A is a linear operator in this space
and $f(t)$ is an almost-periodic function $-\infty < t < \infty \rightarrow \mathbb{R}^n$; a classical statement
[3], establishes the equivalence between the solutions $u(t)$, $-\infty < t < \infty \rightarrow \mathbb{R}^n$

which are bounded and those which are almost-periodic. A similar result can be also demonstrated for various partial differential equations (of the evolution type, like $u_{tt}(x,t) = \Delta u(x,t) + f(x,t)$ in a bounded domain, or $u_t(x,t) = \Delta u(x,t) + f(x,t)$ in the whole space) when almost-periodicity is considered with respect to the "time" variable, as an L^2 -almost-periodicity) (see [2]). In this paper, we prove a variant to a conjecture of us in the paper [7]; precisely, the above explained equivalence between L^2 -bounded and L^2 -almost-periodic solutions is true also for a (parabolic) second order partial differential equation of the form

$$(0.1) \quad \frac{\partial u}{\partial t}(x,t) = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{\ell=1}^n a_{\ell} \frac{\partial u}{\partial x_{\ell}} + f(x,t)$$

where a_{ij} are constant complex coefficients, a_{ℓ} are real constant coefficients, there is a constant $C_0 > 0$ with property that for all $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$,

$$\operatorname{Re} \sum_{j,k=1}^n a_{ij} \delta_j \delta_k \geq C_0 |\delta|^2, \quad \forall (\delta_1, \dots, \delta_n) \in \mathbb{R}^n \text{ and } f(x,t) \text{ is an } L^2\text{-almost-}$$

periodic "input". We use the method of "concrete" orthogonal projections as in our previous paper [6].

§ 1. The study of solutions of (0.1) which are square-integrable over $\mathbb{R}^n$ space as functions of the (x) -variable is best done in the framework of $L^2(\mathbb{R}^n)$ -valued functions. ⁽¹⁾

Let us consider a vector-function $f(t)$, $-\infty < t < \infty \rightarrow L^2(\mathbb{R}^n)$ which is (strongly) continuous and (strongly) almost-periodic (i.e., in Bochner's sense) (see [2] for definitions). Let also $u(t)$, $-\infty < t < \infty \rightarrow L^2(\mathbb{R}^n)$ be a weakly continuously differentiable function (i.e. it has an weakly continuous weak derivative $u'(t)$; this implies strong continuity of $u(t)$ (see for example [4], ex. 39.2, page 388).

⁽¹⁾ Here $L^2(\mathbb{R}^n)$ is the space of square integrable complex valued functions on $\mathbb{R}^n$.

Let be $S(\mathbb{R}^n)$ the space of rapidly decreasing C^∞ -functions, and let us assume the validity of the following integral identity: (where $(\cdot, \cdot)$ will mean the $L^2(\mathbb{R}^n)$ -scalar product for complex-valued functions).

$$(1.1) \quad \frac{d}{dt}(u(t), \varphi)_{L^2} = \left(u(t), - \sum_{\ell, k=1}^n a_{\ell k} \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k} - \sum_{\ell=1}^n a_\ell \frac{\partial \varphi}{\partial x_\ell} \right)_{L^2} + (f(t), \varphi)_{L^2}$$

for every "test-functions" $\varphi(x) \in S(\mathbb{R}^n)$.

Let us remark that (1.1) is verified also for all $\varphi(x) \in H^2(\mathbb{R}^n)$, the space of functions in $L^2(\mathbb{R}^n)$ having first and second order partial weak derivatives φ_{x_ℓ} , $\varphi_{x_\ell x_k}$ again in $L^2(\mathbb{R}^n)$ (see [4]); this is easily seen, because of the density of $S(\mathbb{R}^n)$ into $H^2(\mathbb{R}^n)$ ([4], Prop. 24.9, page 216).

Let us consider now the Fourier-Plancherel operator on L^2 , F which is unitary and has inverse F^{-1} . Let us denote $U(t) = (Fu)(t)$, $F(t) = (Ff)(t)$. It is easily seen that $F(t)$ is again an L^2 -continuous and almost-periodic function and $U(t)$ is L^2 -weakly continuously differentiable. In fact, the weak derivative of U equals Fu' , where u' is the weak derivative of u .

Let us apply now Plancherel's formula, for any $\varphi \in H^2(\mathbb{R}^n)$: we get

$$(1.2) \quad (u(t), \varphi)_{L^2} = (U(t), F\varphi)_{L^2}, \quad (f(t), \varphi)_{L^2} = (F(t), F\varphi)_{L^2}$$

and also

$$(1.3) \quad \begin{aligned} & \left(u(t), - \sum_{\ell, k=1}^n a_{\ell k} \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k} - \sum_{\ell=1}^n a_\ell \frac{\partial \varphi}{\partial x_\ell} \right)_{L^2} \\ &= \left(U(t), F \left(- \sum_{\ell, k=1}^n a_{\ell k} \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k} - \sum_{\ell=1}^n a_\ell \frac{\partial \varphi}{\partial x_\ell} \right) \right)_{L^2} \end{aligned}$$

Let us remember that if we denote $(F\varphi)(s) = \varphi(s)$, $s = (s_1, \dots, s_n)$; then $F\left(\frac{\partial \varphi}{\partial x_\ell}\right) = i s_\ell \varphi(s)$, $i = \sqrt{-1}$; and $F\left(\frac{\partial^2 \varphi}{\partial x_\ell \partial x_k}\right) = -s_\ell s_k \varphi(s)$. Consider the polynomial of n real variables,

$$L(\delta_1, \dots, \delta_n) = \sum_{\ell, k=1}^n a_{\ell k} \delta_{\ell} \delta_k - i \sum_{\ell=1}^n a_{\ell} \delta_{\ell}$$

and remark that, for $\phi \in H^2$, $L(\delta) \phi(\delta) \in L^2$. Then (1.3) equals $(U(t), M_L \phi)$ where M_L is the multiplication operator by the polynomial $L(\delta)$.

Remember also that operator F maps $H^2(\mathbb{R}^n)$ onto the space $\hat{H}^2$ of functions $\phi \in L^2$, such that $\delta_{\ell} \phi(\delta) \in L^2$, $\forall \ell = 1, 2, \dots, n$, and $\delta_{\ell} \delta_k \phi(\delta) \in L^2$, $\forall \ell, k = 1, 2, \dots, n$. Consequently (1.1) is equivalent to the identity

$$(1.4) \quad \frac{d}{dt} (U(t), \phi)_{L^2} = (U(t), M_L \phi)_{L^2} + (F(t), \phi)_{L^2},$$

for any real t and $\forall \phi \in \hat{H}^2$.

§2. Consider a sequence of functions $\chi_p(\delta)$, defined as follows:

$$\chi_p(\delta) = 1 \text{ if } \frac{1}{p+1} \leq |\delta| \leq \frac{1}{p}, \quad \chi_p(\delta) = 0 \text{ otherwise } (\delta = (\delta_1, \dots, \delta_n)),$$

$$|\delta|^2 = \delta_1^2 + \dots + \delta_n^2, \quad p = 1, 2, \dots$$

To any such sequence one associates a sequence of bounded linear (multiplication) operators on $L^2(\mathbb{R}^n)$, defined by the relation $(H_p \psi)(\delta) = \chi_p(\delta) \psi(\delta)$, $\forall \psi \in L^2$, $p = 1, 2, \dots$

Consider then the sequence of L^2 -valued functions $F_p(t) = (H_p F)(t)$, $p = 1, 2, \dots$, $U_p(t) = (H_p U)(t)$, $p = 1, 2, \dots$, $-\infty < t < \infty$.

We shall prove the following

Lemma 2.1. The functions $U_p(t)$ are strongly continuously differentiable on the real line, and the relations

$$(2.1) \quad \frac{d U_p(t)}{dt} = (M_L H_p) U_p(t) + F_p(t), \quad -\infty < t < \infty, \quad p = 1, 2, \dots$$

are verified (in $L^2(\mathbb{R}^n)$).

(Here M_L is the multiplication operator by the polynomial

$$L(\delta) = L(\delta) = \sum_{\ell, k=1}^n \bar{a}_{\ell k} \delta_{\ell k} + i \sum_{\ell=1}^n a_{\ell} \delta_{\ell} ; \text{ the relation (2.1) makes sense because}$$

H_p maps $L^2(\mathbb{R}^n)$ on the subspace $L_p^2(\mathbb{R}^n)$ consisting of functions which are null for $0 \leq |\delta| \leq \frac{1}{p+1}$ and for $|\delta| > \frac{1}{p}$, and M_L is obviously defined on the whole of L_p^2 as a linear continuous map .)

Proof of Lemma 2.1. Let us use (1.4): If $\phi \in L^2(\mathbb{R}^n)$, then $X_p \phi \in L_p^2(\mathbb{R}^n)$ and $L_p^2(\mathbb{R}^n) \subset \hat{H}^2$; we can write

$$\frac{d}{dt} (U(t), X_p \phi)_{L^2} = (U(t), M_L(X_p \phi))_{L^2} + (F(t), X_p \phi)_{L^2} .$$

Remark now that $(\psi, X_p \phi)_{L^2} = (X_p \psi, \phi)_{L^2}$, $\forall \psi, \phi \in L^2$, so that

$$(U(t), X_p \phi) = (X_p U(t), \phi) = (H_p U, \phi) . \text{ Also}$$

$$M_L(X_p \phi) = M_L(H_p \phi) = H_p M_L H_p \phi ,$$

because $M_L H_p$ maps $L^2(\mathbb{R}^n)$ in $L_p^2(\mathbb{R}^n)$ and H_p is the identity operator on $L_p^2(\mathbb{R}^n)$.

These remarks imply the equality

$$\frac{d}{dt} (H_p U, \phi) = (H_p U(t), M_L H_p \phi) + (H_p F, \phi) .$$

Remark also that, if $\psi \in L_p^2$ and $\phi \in L^2$, it is

$$\begin{aligned} (\psi, M_L H_p \phi) &= \int_{\mathbb{R}^n} \psi(s) \overline{L(\delta)} X_p(s) \overline{\phi}(s) ds \\ &= \int_{\mathbb{R}^n} L(\delta) X_p(s) \psi(s) \overline{\phi}(s) ds \\ &= (M_L H_p \psi, \phi)_{L^2} . \end{aligned}$$

We obtained the equality

$$\frac{d}{dt} (U_p(t), \phi) = (M_L H_p U_p(t), \phi) + (F_p(t), \phi) , \quad \forall \phi \in L^2 .$$

Now, $U_p(t)$, like $u(t)$ and $U(t)$ is strongly continuous; also, if U' is the weak (L^2) derivative of $U(t)$, then $H_p U'$ will be the weak derivative of $H_p U$, as easily seen. Hence U'_p (weak L^2 -derivative) exists and equals $H_p U'$. The above written equality becomes,

$$(U'_p(t) - M_L H_p U_p(t) - F_p(t), \phi)_{L^2} = 0, \quad \forall \phi \in L^2,$$

so that

$$U'_p(t) = M_L H_p U_p(t) + F_p(t), \quad \forall t \in \mathbb{R} \text{ (in } L^2 \text{ or } L^2_p).$$

Hence, the weak derivative $U'_p(t)$ is strongly continuous, and this implies that the weak derivative is also a strong derivative. This ends the proof of Lemma 2.1.

§3. Our first remark here will be the relation

$$\sup_{-\infty < t < \infty} \|U_p(t)\|_{L^2(\mathbb{R}^n)} \leq L < \infty, \quad p = 1, 2, \dots$$

which is an obvious corollary of the definition: $U_p(t) = (H_p U)(t)$, of the estimate $\|U_p(t)\|_{L^2} \leq \|U(t)\|_{L^2}$ and of the

Main hypothesis. The function $u(t)$ in (1.1) is bounded on the real line.

We can say now, in view of Lemma 2.1, that $U_p(t)$ is a bounded solution on the real line of the differential equation (2.1), $\forall p = 1, 2, \dots$ where obviously, all functions $F_p(t)$ are $L^2(\mathbb{R}^n)$ bounded and almost-periodic.

We also need an estimate for the operator $M_L H_p$ appearing in the right-hand side of (2.1), when acting from $L^2(\mathbb{R}^n)$ into itself.

Taking an arbitrary function $\phi(\delta) \in L^2(\mathbb{R}^n)$, we obtain

$$(M_L H_p \phi)(\delta) = L(\delta) \chi_p(s) \phi(s)$$

and consequently

$$\|M_{L^p}^H \phi\|_{L^2(\mathbb{R}^n)} = \left(\int_{\frac{1}{p+1} \leq |\delta| \leq \frac{1}{p}} |L(\delta) \phi(\delta)|^2 d\delta \right)^{\frac{1}{2}}.$$

On the other hand

$$L(\delta) \phi(\delta) = \left(\sum_{\ell,k=1}^n \bar{a}_{\ell k} \delta_{\ell} \delta_k \right) \phi(\delta) + i \left(\sum_{\ell=1}^n a_{\ell} \delta_{\ell} \right) \phi(\delta)$$

$$|L(\delta) \phi(\delta)| \leq \sum_{\ell,k=1}^n |a_{\ell k}| |\delta_{\ell}| |\delta_k| |\phi(\delta)| + \sum_{\ell=1}^n |a_{\ell}| |\delta_{\ell}| |\phi(\delta)|.$$

For $\ell = 1, 2, \dots, n$, $|\delta_{\ell}| \leq |\delta| = (\delta_1^2 + \dots + \delta_n^2)^{\frac{1}{2}}$ so that

$$|L(\delta) \phi(\delta)| \leq |\delta|^2 |\phi(\delta)| \sum_{\ell,k=1}^n |a_{\ell k}| + |\delta| |\phi(\delta)| \sum_{\ell=1}^n |a_{\ell}| = (A_1 |\delta|^2 + A_2 |\delta|) |\phi(\delta)|$$

where $A_1 = \sum_{\ell,k=1}^n |a_{\ell k}|$, $A_2 = \sum_{\ell=1}^n |a_{\ell}|$.

In the above integral we always have $|\delta| \leq \frac{1}{p}$, so that

$$|L(\delta) \phi(\delta)|^2 \leq \left(\frac{A_1}{p^2} + \frac{A_2}{p} \right)^2 |\phi(\delta)|^2$$

and

$$\left(\int_{\frac{1}{p+1} \leq |\delta| \leq \frac{1}{p}} |L(\delta) \phi(\delta)|^2 d\delta \right)^{\frac{1}{2}} \leq \left(\frac{A_1}{p^2} + \frac{A_2}{p} \right) \|\phi\|_{L^2(\mathbb{R}^n)}.$$

The operator $M_{L^p}^H$ is linear continuous in $L^2(\mathbb{R}^n)$; its operator norm is less

than $\frac{A_1}{p^2} + \frac{A_2}{p}$, $\forall p = 1, 2, \dots$

We prove next the following

Lemma 3.1. The differential equation (2.1) has one and only one solution which is bounded over the real line.

i) To prove uniqueness of bounded solutions it is sufficient to consider the homogeneous equation

$$\frac{dV}{dt} = (M_{LH_p})V, \quad -\infty < t < \infty, \quad \text{in } L^2(\mathbb{R}^n).$$

Due to boundedness of the operator M_{LH_p} , we see that, $\forall t_0 \in \mathbb{R}$ and $\forall t \in \mathbb{R}$,

$$V(t) = \exp(M_{LH_p})(t-t_0)V(t_0)$$

where, as usual

$$\exp(\xi M_{LH_p}) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} (M_{LH_p})^n, \quad \forall \xi \in \mathbb{R}.$$

We need a good estimate for the norm of this exponential operator; we establish first, by an easy induction argument, the representation formula

$$(M_{LH_p})^n \phi(s) = \begin{cases} L^n(s) \phi(s), & \text{for } \frac{1}{p+1} \leq |s| \leq \frac{1}{p} \\ 0, & \text{otherwise } \forall \phi \in L^2(\mathbb{R}^n), \forall s \in \mathbb{R}^n \end{cases}$$

(in fact: $[(M_{LH_p})\phi](s) = \{L(s) \chi_p(s) \phi(s)\}$; consider $n+1$;

$$\begin{aligned} [(M_{LH_p})^{n+1}\phi](s) &= [(M_{LH_p})(M_{LH_p})^n\phi](s) = (M_{LH_p})(L^n(s) \chi_p(s) \phi(s)) \\ &= L(s) \chi_p(s) \cdot (L^n(s) \chi_p(s) \phi(s)) = L^{n+1}(s) \chi_p(s) \phi(s), \quad \forall s \in \mathbb{R}^n. \end{aligned}$$

Hence

$$\begin{aligned} [\exp(\xi M_{LH_p})\phi](s) &= \sum_{n=0}^{\infty} \frac{\xi^n}{n!} L^n(s) \chi_p(s) \phi(s), \quad \forall s \in \mathbb{R}^n \\ &= \left(\sum_{n=0}^{\infty} \frac{\xi^n}{n!} L^n(s) \right) \chi_p(s) \phi(s) = \exp(\xi L(s)) \chi_p(s) \phi(s), \quad \forall s \in \mathbb{R}^n. \end{aligned}$$

It follows that

$$\|\exp(\xi M_{LH_p})\phi\|_{L^2} = \left\{ \int_{\frac{1}{p+1} \leq |s| \leq \frac{1}{p}} |e^{\xi L(s)} \phi(s)|^2 ds \right\}^{\frac{1}{2}}, \quad \forall \xi \in \mathbb{R}.$$

We see that

$$|e^{\xi L(s)}| = \exp(\xi \operatorname{Re} L(s)) = e^{\xi \operatorname{Re} \sum_{k=1}^n a_k s_k}.$$

If we estimate for $\xi < 0$, we obtain, in view of $\operatorname{Re} \sum_{\ell, k=1}^n a_{\ell k} s_{\ell} s_k \geq C_0 |\Delta|^2$, the inequality

$$|e^{\xi L(s)}| \leq e^{C_0 \xi |\Delta|^2}, \quad \xi < 0, \quad \Delta \in \mathbb{R}^n \quad (C_0 > 0)$$

so that

$$\begin{aligned} \|\exp(\xi M_{L^H P}) \phi\|_{L^2} &= \left\{ \int_{\frac{1}{p+1} \leq |s| \leq \frac{1}{p}} e^{2\xi \operatorname{Re} L(s)} |\phi(s)|^2 ds \right\}^{\frac{1}{2}} \\ &\quad \left(\text{because } |s|^2 > \frac{1}{(p+1)^2}, \quad \xi < 0 \Rightarrow \xi |s|^2 \leq \frac{\xi}{(p+1)^2} \right) \\ &\leq \left\{ \int_{\frac{1}{p+1} \leq |s| \leq \frac{1}{p}} e^{2C_0 \xi |\Delta|^2} |\phi(s)|^2 ds \right\}^{\frac{1}{2}} \\ &\leq \left\{ \int_{\frac{1}{p+1} \leq |s| \leq \frac{1}{p}} e^{2C_0 \frac{\xi}{(p+1)^2}} |\phi(s)|^2 ds \right\}^{\frac{1}{2}} \leq e^{C_0 \frac{\xi}{(p+1)^2}} \|\phi\|_{L^2}, \end{aligned}$$

$$\forall \xi < 0, \quad \forall p = 1, 2, \dots, \quad \forall \phi \in L^2.$$

We use now formula

$$V(t) = \exp[(M_{L^H P})(t-t_0)] V(t_0), \quad \text{for } t < t_0,$$

so that we have also the estimate

$$\|V(t)\|_{L^2} \leq e^{\frac{C_0}{(p+1)^2} (t-t_0)} \|V(t_0)\|_{L^2}, \quad \forall t < t_0.$$

We assume that $V(t)$ is a bounded solution on the line: $\sup_{t \in \mathbb{R}} \|V(t)\| = L < \infty$;

hence, taking $t_0 = n$

$$\|V(t)\|_{L^2} \leq L e^{\frac{C_0}{(p+1)^2} t} e^{-n \frac{C_0}{(p+1)^2}}, \quad \forall n = 1, 2, \dots$$

when $n \rightarrow \infty$ we deduce $\|V(t)\|_{L^2} = 0$; t was arbitrarily chosen

ii) The existence of the (unique) bounded solution of the equation

$$\frac{dU_p}{dt} = (M_L H_p) U_p + F_p$$

is now an immediate application of Theorem 1 in our paper [5]. (This solution was given as the infinite series

$$U_p(t) = \sum_{n=1}^{\infty} V_{p,n}(t)$$

where

$$V_{p,n}(t) = - \int_{-n}^{-n+1} \exp(M_L H_p \zeta) F_p(t-\zeta) d\zeta .$$

This ends proof of Lemma 3.1.

Remark. As follows from [5], $U_p(t)$ is, $\forall p$, an L^2 -almost periodic function; then also $(M_L H_p)U_p(t)$ will be almost-periodic, $\forall p = 1, 2, \dots$

§4. Consider now the infinite series $\sum_{p=1}^{\infty} U_p(t)$; we shall see below that it is pointwise convergent in the $L^2(\mathbb{R}^n)$ -norm to the function XU ; here the map $\phi \rightarrow X\phi$ is the multiplication operator $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ defined by the formula $\phi(s) \rightarrow X(s) \phi(s)$ where $X(s)$ is the characteristic function of the unit sphere.

In fact,

$$\|XU - \sum_{\ell=1}^p U_{\ell}\|_{L^2}^2 = \left\| \left(X - \sum_{\ell=1}^p H_{\ell} \right) U \right\|_{L^2}^2 ;$$

for any real t .

Then

$$\begin{aligned} \left\| \left(X - \sum_{\ell=1}^p H_{\ell} \right) U \right\|_{L^2}^2 &= \left\{ \int_{\mathbb{R}^n} \left| X(s) - \sum_{\ell=1}^p X_{\ell}(s) U(s, t) \right|^2 ds \right\}^{\frac{1}{2}} \\ &= \left(\int_{0 \leq |s| \leq \frac{1}{p+1}} |U(s, t)|^2 ds \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } p \rightarrow \infty \end{aligned}$$

by the absolute continuity property of L^2 functions.

Let us consider now the infinite series $\sum_{p=1}^{\infty} \frac{dU_p}{dt}$; we shall see that this series is uniformly convergent over $-\infty < t < +\infty$, again in $L^2(\mathbb{R}^n)$ sense. In view of the equality expressed in (2.1) it suffices to prove this uniform convergence for the two series: $\sum_{p=1}^{\infty} F_p(t)$ and $\sum_{p=1}^{\infty} (M_{L_p}^H)U_p$, again in $L^2(\mathbb{R}^n)$.

The first series is again pointwise convergent to the function XF , which is obviously L^2 -almost-periodic. It has consequently a relatively compact range, and this implies uniform convergence of $\sum_{p=1}^{\infty} F_p(t)$ to $(XF)(t)$ in the well-known way.

In order to prove uniform convergence of the series $\sum_{p=1}^{\infty} (M_{L_p}^H)U_p(t)$, we remark first the orthogonality relations

$$(M_{L_p}^H U_p(t), M_{L_q}^H U_q(t))_{L^2} = 0, \quad \forall p \neq q$$

so that it will suffice to prove uniform convergence of the series

$\sum_{p=1}^{\infty} \|(M_{L_p}^H)U_p(t)\|_{L^2}^2$. But here we can use the estimate obtained in §3, that is

$$\|M_{L_p}^H \phi\|_{L^2} \leq \left(\frac{A_1}{p^2} + \frac{A_2}{p} \right) \|\phi\|_{L^2}, \quad \forall \phi \in L^2;$$

we have

$$\|(M_{L_p}^H)U_p(t)\|^2 \leq \left(\frac{A_1}{p^2} + \frac{A_2}{p} \right)^2 \|U_p(t)\|^2 \leq \left(\frac{A_1}{p^2} + \frac{A_2}{p} \right)^2 L,$$

in view of the main assumption about boundedness of $U(t)$ on the real line.

A corollary of this uniform convergence is the L^2 -almost-periodicity of the sum of $\sum_{p=1}^{\infty} \frac{dU_p}{dt}$; furthermore, let us remember that $(XU)(t) = \sum_{p=1}^{\infty} U_p(t)$, in

$L^2(\mathbb{R}^n)$ pointwise convergence. Now, it is easy to see that $(XU)(t)$ has a strong

$L^2(\mathbb{R}^n)$ -derivative which equals $\sum_{p=1}^{\infty} \frac{dU_p}{dt}$ - an almost-periodic function. As

$(XU)(t)$ is L^2 -bounded on the real line, it follows (by Amerio's theorem [1]) that XU is L^2 -almost-periodic. Next, we must prove the L^2 -almost-periodicity of $(I-X)U$; this is done (as in our paper [6]) so that we can conclude about almost-periodicity of bounded solution of (1.1) when the right-hand side is almost-periodic.

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