

ON THE LOWER ORDER OF ENTIRE FUNCTIONS DEFINED BY L-DIRICHLETIAN ELEMENTS

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RÉSUMÉ

Les auteurs se proposent d'étendre à une classe de fonctions entières non bornées dans chaque demi-plan $P_z = \{s \in \mathbb{C} \mid \operatorname{Re} s > \operatorname{Re} z\}$, pour z décrivant $\mathbb{C}$, la notion d'ordre inférieur, au sens de Ritt-Rahman, des fonctions entières définies par des éléments dirichletiens classiques. Ici, les séries de définition des fonctions considérées sont du type $\sum P_n(s) \exp(-s\lambda_n)$ où les $P_n(s)$ sont des polynômes tayloriens et où la suite (λ_n) est une D-suite. La méthode utilisée est analogue, dans son principe, à celle utilisée par Maurice Blambert et Marcel Berland pour définir l'ordre-R de telles fonctions.

ABSTRACT

In this work, functions defined by the series $\sum_{n=1}^{\infty} P_n(s) \exp(-s\lambda_n)$, where (λ_n) is a D-sequence and $P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j$ are considered. As these functions are not bounded in the half plane, it is not possible to define the lower order (in the sense of Ritt-Rahman) directly. The purpose of this paper is to define the lower order for these function, by making use of functions defined by associated intermediate series. In the particular case when the polynomial $P_n(s)$ reduces to a complex number, then we get the well known definition introduced by

Q.I. Rahman for the lower-order of entire functions defined by Dirichlet series.

Let (λ_n) be a D-sequence, (P_n) be a sequence of complex valued polynomials, that is

$$\forall_{n \in \mathbb{N} - \{0\}} P_n : \mathbb{C} \ni s \rightarrow P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j \quad \text{with } a_{nm_n} \neq 0$$

and $s = \sigma + i\tau$, $(\sigma, \tau) \in \mathbb{R}^2$, consider the L-dirichletian elements

$$\{\varphi\} : \sum_{n=1}^{\infty} P_n(s) \exp(-s\lambda_n)$$

and the space of elements of the type

$$\{\varphi_{\tau_0}\} : \sum_{n=1}^{\infty} P_n(\sigma + i\tau_0) \exp(-s\lambda_n)$$

indexed by τ_0 on $\mathbb{R}$. Let $\beta = \limsup \{m_n/\lambda_n / n \in \mathbb{N} - \{0\}\}$,

$\beta^* = \sup \{m_n/\lambda_n / n \in \mathbb{N} - \{0\}\}$, $A_n = \max \{|a_{nj}| / j \in \{0, 1, \dots, m_n\}\}$ and

$$L = \lim \left\{ \frac{\log n}{\lambda_n} / n \in \mathbb{N} - \{0\} \right\}.$$

Consider the associated dirichletian element

$$\{\psi\} : \sum_{n=1}^{\infty} A_n \exp(-\lambda_n s)$$

whose coefficients are strictly positive and denote by σ_c^ψ the abscissa of convergence of $\{\psi\}$. We can easily show that if $\sigma_c^\psi = -\infty$, then $\{\varphi_{\tau_0}\}$ converges absolutely on $\mathbb{C}$ for any arbitrary τ_0 in $\mathbb{R}$ if $\beta < \infty$.

In what follows, we always assume that $\beta < \infty$, $L < \infty$, and $\sigma_c^\psi = -\infty$. The fact that $\beta < \infty$ implies that $\beta^* < \infty$ and reciprocally that $\beta^* < \infty$ implies that $\beta < \infty$.

Let us define the following quantities. For each $\sigma \in \mathbb{R}$

$$M(\sigma; \varphi_{\tau_0}) = \sup \{ |\varphi_{\tau_0}(s')| / \operatorname{Re} s' \geq \sigma \wedge \tau' \in \mathbb{R} \},$$

$$m(\sigma; \varphi_{\tau_0}) = \sup \{ |\varphi_{\tau_0}(s')| / \operatorname{Re} s' = \sigma \wedge \tau' \in \mathbb{R} \},$$

$$\mu(\sigma; \varphi_{\tau_0}) = \sup \{ |P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) / n \in \mathbb{N} - \{0\} \}$$

$$m_{n'}(\sigma; \varphi_{\tau_0}) = \sup \{ |\varphi_{\tau_0}(s) - \sum_{l=1}^{n'-1} P_n(\sigma + i\tau_0) \exp(-\lambda_n s)| / \tau \in \mathbb{R} \}$$

and

$$\mu_{n'}(\sigma; \varphi_{\tau_0}) = \sup \{ |P_n(\sigma + i\tau_0)| \exp(-\sigma \lambda_n) / n \geq n' \}$$

where $\varphi_{\tau_0}(s)$ denotes the sum of the series $\{\varphi_{\tau_0}\}$ at the point $s \in \mathbb{C}$. For each $\sigma \in \mathbb{R}$, these quantities defined above are finite. Further, let

$$\lambda_M^{\varphi_{\tau_0}} = \liminf_{\sigma \rightarrow -\infty} \frac{\log \log M(\sigma; \varphi_{\tau_0})}{-\sigma}$$

similarly $\lambda_m^{\varphi_{\tau_0}}$, $\lambda_\mu^{\varphi_{\tau_0}}$, $\lambda_{m_{n'}}^{\varphi_{\tau_0}}$ and $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}}$ are defined with $m(\sigma; \varphi_{\tau_0})$, $\mu(\sigma; \varphi_{\tau_0})$, $m_{n'}(\sigma; \varphi_{\tau_0})$ and $\mu_{n'}(\sigma; \varphi_{\tau_0})$ respectively in the place of $M(\sigma; \varphi_{\tau_0})$. Also,

$M(\sigma; \psi)$, $m(\sigma; \psi)$, $\mu(\sigma; \psi)$, $m_{n'}(\sigma; \psi)$, $\mu_{n'}(\sigma; \psi)$, λ_M^ψ , λ_m^ψ , λ_μ^ψ , $\lambda_{m_{n'}}^\psi$ and $\lambda_{\mu_{n'}}^\psi$

are defined in a similar manner with ψ in the place φ_{τ_0} . It is trivial that

$$\lambda_M^\psi = \lambda_m^\psi \text{ and } \lambda_M^{\varphi_{\tau_0}} \geq \lambda_m^{\varphi_{\tau_0}}.$$

Remark 1. We have $\forall \sigma \in \mathbb{R} \quad \forall n \in \mathbb{N} - \{0\} \quad |P_n(\sigma + i\tau_0)| \exp(-\sigma \lambda_n) \leq \mu(\sigma; \varphi_{\tau_0})$, and hence $\mu(\sigma; \varphi_{\tau_0}) \rightarrow \infty$ as $\sigma \rightarrow -\infty$, which shows that $\lambda_\mu^{\varphi_{\tau_0}} \geq 0$.

Let us state a theorem due to M. Blambert and M. Berland [1], which we will use later.

Theorem A. When $\sigma_C^\psi = -\infty$, we have

$$\forall \sigma \in \mathbb{R} \quad \forall n \in \mathbb{N} - \{0\} \quad P_n(\sigma + i\tau_0) \exp(-\lambda_n \sigma) = \lim_{\tau_2 \rightarrow \infty} \left\{ \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} \varphi_{\tau_0}(\sigma + i\tau) \exp(i\tau \lambda_n) d\tau \right\}$$

where τ_1 is any arbitrary real number.

Remark 2. From this theorem, we deduce immediately that

$$\forall \sigma \in \mathbb{R} \quad \forall n \in \mathbb{N} - \{0\} \quad |P_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) \leq m(\sigma; \varphi_{\tau_0}) .$$

Further, we can show that there exists no sequence (σ_j) , σ_j negative and $\lim_{j \rightarrow \infty} \sigma_j = -\infty$, such that the map $\mathbb{R} \ni \sigma \rightarrow m(\sigma; \varphi_{\tau_0})$ is of exponential type on (σ_j)

(that is we can show that there cannot exist a constant $\alpha > 0$ such that

$$\forall_j m(\sigma_j; \varphi_{\tau_0}) \leq \exp(-\alpha\sigma_j) .$$

On the contrary, let us assume that there exists such

a constant α associated to a certain sequence of the space precised, such that the above condition is satisfied. Then we have

$$\forall n \in \mathbb{N} - \{0\} \quad \forall_j |P_n(\sigma_j + i\tau_0)| \leq \exp \sigma_j (\lambda_n - \alpha) ,$$

from which it follows that, for each n , with $\lambda_n > \alpha$, $\lim_{j \rightarrow \infty} |P_n(\sigma_j + i\tau_0)| = 0$.

But we know that $|P_n(\sigma + i\tau_0)| \rightarrow \infty$ as $\sigma \rightarrow \infty$, which shows that our assumption leads to a contradiction thus proving the validity of our assertion.

Also we need the following lemmas to prove our theorems.

Lemma 1. $\forall n' \in \mathbb{N} - \{0\}$ we have $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} = \lambda_{\mu}^{\varphi_{\tau_0}}$.

Proof. Put $\mu_{n'}^0(\sigma; \varphi_{\tau_0}) = \text{Max} \{ |P_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) / n \in \{1, \dots, n'-1\} \}$.

Then it is trivial that $\lim_{\sigma \rightarrow -\infty} \frac{\log \log \mu_{n'}^0(\sigma; \varphi_{\tau_0})}{-\sigma} = 0$. Since the mapping for

any $n' \in \mathbb{N} - \{0\}$,

$$\mathbb{R} \ni \sigma \rightarrow \mu_{n'}(\sigma; \varphi_{\tau_0}) = \sup \{ |P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) / n' \geq n \}$$

is not of exponential type for any sequence (σ_j) , σ_j negative and

$\lim_{j \rightarrow \infty} \sigma_j = -\infty$, and also since

$$\mu(\sigma; \varphi_{\tau_0}) = \text{Max} \{ \mu_{n'}(\sigma; \varphi_{\tau_0}); \mu_{n'}^0(\sigma; \varphi_{\tau_0}) \} ,$$

we have $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} = \lambda_{\mu}^{\varphi_{\tau_0}}$.

Lemma 2. $\forall n' \in \mathbb{N} - \{0\}$ we have $\lambda_{m_{n'}}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}}$.

Proof. Let λ^0 be a constant such that $\lambda^0 > \lambda_{n'-1}$. Since

$$m_{n'}(\sigma; \varphi_{\tau_0}) = \text{Max} \{ |\varphi_{\tau_0}(s) - \sum_{l=1}^{n'-1} P_n(\sigma + i\tau_0) \exp(-s\lambda_n) | / \tau \in \mathbb{R} \} ,$$

$\exists \sigma_1 > 0 \forall \sigma < -\sigma_1 \quad m(\sigma; \varphi_{\tau_0}) - \exp(-\sigma\lambda^0) < m_{n'}(\sigma; \varphi_{\tau_0}) < m(\sigma; \varphi_{\tau_0}) + \exp(-\sigma\lambda^0)$; hence from

Remark 2, we have $\lambda_{m_{n'}}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}}$.

Theorem 1. When $\sigma_c^\psi = -\infty$, $\beta < \infty$ and $L < \infty$, we have

$$\forall \tau_0 \in \mathbb{R} \quad 0 \leq \lambda_{\mu}^{\varphi_{\tau_0}} \leq \lambda_{\mu}^{\psi} = \lambda_M^{\psi} .$$

Proof. We have $\forall \beta' > \beta \quad \exists n' (= n_{\beta'}) \quad \forall n \geq n' \quad \frac{m_n}{\lambda_n} < \beta'$, and hence for $n \geq n'$,

σ negative

$$|P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) < A_n \exp\{-\sigma\lambda_n [1 + \frac{\beta'}{|\sigma|} \log(1 + |\sigma| + |\tau_0|)]\} ;$$

for $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that

$$\forall \sigma < -\sigma_\varepsilon \quad |P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) < A_n \exp\{-\sigma\lambda_n(1 + \beta'\varepsilon)\}$$

which gives

$$\mu_{n'}(\sigma; \varphi_{\tau_0}) \leq \mu_{n'}(\sigma(1 + \beta'\varepsilon); \psi)$$

and

$$\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} \leq \lambda_{\mu_{n'}}^{\psi} (1 + \beta'\varepsilon) .$$

As ε is arbitrary, we have $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} \leq \lambda_{\mu_{n'}}^{\psi}$. Using lemma 1, we have

$$\lambda_{\mu}^{\varphi_{\tau_0}} = \lambda_{\mu_{n'}}^{\varphi_{\tau_0}} \leq \lambda_{\mu_{n'}}^{\psi} \quad (1)$$

Let $L' > L$. Then

$$\forall_{n \geq n'} \quad A_n \exp(-\sigma \lambda_n) < \psi_{n'}(\sigma) = \sum_{j \geq n'} A_j \exp\{-(\sigma - L')\lambda_j\} \exp\{-L'\lambda_j\},$$

and hence

$$\mu_{n'}(\sigma; \psi) < \psi_{n'}(\sigma) < \mu_{n'}(\sigma - L'; \psi) C_{n'}(L'),$$

where $C_{n'}(L') = \sum_{j=n'}^{\infty} \exp(-L'\lambda_j)$ and $\psi_{n'}(\sigma) = \sum_{n=n'}^{\infty} A_n \exp(-\lambda_n \sigma)$, which gives

$$\forall_{n' \in \mathbb{N} - \{0\}} \quad \lambda_{\mu_{n'}}^{\psi} = \lambda_M^{\psi_{n'}} \quad (\text{when } n' = 1, \quad \lambda_{\mu}^{\psi} = \lambda_M^{\psi}). \quad (2)$$

As it is trivial that $\lambda_M^{\psi_{n'}} = \lambda_M^{\psi}$, we have from (1) and (2) that

$$\forall_{\tau_0 \in \mathbb{R}} \quad 0 \leq \lambda_{\mu}^{\varphi_{\tau_0}} = \lambda_{\mu_{n'}}^{\varphi_{\tau_0}} \leq \lambda_{\mu_{n'}}^{\psi} = \lambda_M^{\psi_{n'}} = \lambda_M^{\psi}$$

thus proving the theorem.

Remark 3. This theorem can also be proved in the following way. Since $\beta < \infty$, we have $\beta^* < \infty$. Then for each $n \geq 1$ and $\sigma < 0$ we have

$$|P_n(\sigma + i\tau_0)| \exp(-\sigma \lambda_n) < A_n \exp\{-\sigma \lambda_n (1 + \frac{\beta^*}{|\sigma|} \log(1 + |\sigma| + |\tau_0|))\}$$

and hence for $\sigma < -\sigma_{\varepsilon}$, $\mu(\sigma; \varphi_{\tau_0}) \leq \mu(\sigma(1 + \varepsilon\beta^*); \psi)$; since ε is arbitrary,

we have $\lambda_{\mu}^{\varphi_{\tau_0}} \leq \lambda_{\mu}^{\psi}$.

By an inequality, shown earlier, for $n' = 1$, we have

$$\mu(\sigma; \psi) < \psi(\sigma) < \mu(\sigma - L'; \psi) C(L')$$

where $C(L')$ is the sum of $\sum_n \exp(-L'\lambda_n)$; we have $\lambda_\mu^\psi \leq \lambda_M^\psi$, thereby we get the inequality $\lambda_\mu^{\varphi_{\tau_0}} \leq \lambda_M^\psi$.

Corollary 1. If $\sigma_C^\psi = -\infty$, β and L are finite, and the function

$$\Phi : \mathbb{N} - \{0\} \ni n \rightarrow \Phi(n) = \frac{\log \frac{A_n}{A_{n+1}}}{\lambda_{n+1} - \lambda_n} \quad (3)$$

is a non decreasing function of n for $n > n_0$, then

$$\forall_{\tau_0 \in \mathbb{R}} \quad 0 \leq \lambda_\mu^{\varphi_{\tau_0}} \leq \liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log(A_n^{-1})} \right\}.$$

Under these conditions on L and Φ , Q.I. Rahman [2] proved that

$$\lambda_M^\psi \leq \liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log(A_n^{-1})} \right\},$$

and hence the corollary follows immediately from theorem 1.

Let α_{nj} , $j \in (1, 2, \dots, m_n)$ be the m_n zeros of $P_n(s)$ with the order of multiplicity taken into account, and $\omega_{nj}(s) = \frac{s - \alpha_{nj}}{s - L' - \alpha_{nj}}$ for $L' > L$. Further, let θ be any positive non zero real number and denote by $E_n(\theta)$ the following set indexed by n on $\mathbb{N} - \{0\}$,

$$E_n(\theta) = \left\{ s \in \mathbb{C} / \forall_{j \in (1, 2, \dots, m_n)} |\omega_{nj}(s)| \leq \theta \right\}$$

and $E(\theta) = \bigcap_n E_n(\theta)$, supposed to be non-empty. Let $\theta \in]0, 1]$. If

$(\sigma + i\tau_0) \in E(\theta)$, then

$$\sum_n |P_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) \leq \mu(\sigma - L'; \varphi_{\tau_0}) C(L').$$

Let $\theta \in \mathbb{R}^+ - [0, 1]$. If $(\sigma + i\tau_0) \in E(\theta)$ then

$$\forall_{n \in \mathbb{N} - \{0\}} |P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) < \mu(\sigma - L'; \varphi_{\tau_0}) \exp(-\lambda_n(L' - \beta^* \log \theta)) .$$

Choose L' such that $0 \leq \log \theta < \frac{L' - L}{\beta^*}$. (This implies that $L' > L$.) Then

$\sum_n \exp\{-\lambda_n(L' - \beta^* \log \theta)\}$ converges to $C(L' - \beta^* \log \theta)$, (say). Then

$$\sum_{n \in \mathbb{N} - \{0\}} |P_n(\sigma + i\tau_0)| \exp(-\lambda_n \sigma) \leq \mu(\sigma - L'; \varphi_{\tau_0}) C(L' - \beta^* \log \theta) .$$

Hence in both cases, if $(\sigma + i\tau_0) \in E(\theta)$, we have

$$m(\sigma; \varphi_{\tau_0}) \leq \mu(\sigma - L'; \varphi_{\tau_0}) C_1$$

where $C_1 = C(L')$ or $C_1 = C(L' - \beta^* \log \theta)$ as the case may be. From Remark 2, we have $\mu(\sigma; \varphi_{\tau_0}) \leq m(\sigma; \varphi_{\tau_0})$ and hence

$$\mu(\sigma; \varphi_{\tau_0}) \leq m(\sigma; \varphi_{\tau_0}) \leq \mu(\sigma - L'; \varphi_{\tau_0}) C_1 .$$

Further supposing that $\exists_{\theta \in \mathbb{R}^+ - \{0\}} \exists_{\sigma_0 > 0} \forall_{\sigma < -\sigma_0} (\sigma + i\tau_0) \in E(\theta)$, we have $\lambda_{\mu}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}}$.

By virtue of theorem 1, we have $\lambda_{\mu}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}} \leq \lambda_M^{\psi}$. Hence we have established

Theorem 2. If $\sigma_c^{\psi} = -\infty$, $\beta < \infty$, $L < \infty$ and

$$\exists_{\theta > 0} \exists_{\sigma_0 > 0} \forall_{\sigma < -\sigma_0} (\sigma + i\tau_0) \in E(\theta) \text{ then } \lambda_{\mu}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}} \leq \lambda_M^{\psi} .$$

Remark 4. We can prove this result by considering a certain $\beta' > \beta$ in the place of β^* . In a similar manner, we can show that $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} = \lambda_{m_{n'}}^{\varphi_{\tau_0}}$ where $n' = n_{\beta}$.

Also we have seen that $\lambda_{\mu_{n'}}^{\varphi_{\tau_0}} = \lambda_{\mu}^{\varphi_{\tau_0}}$ and $\lambda_{m_{n'}}^{\varphi_{\tau_0}} = \lambda_m^{\varphi_{\tau_0}}$ and hence the result

follows immediately.

Theorem 3. If $\sigma_c^{\psi} = -\infty$ and $\beta < \infty$ then $\forall_{\tau_0 \in \mathbb{R}} \lambda_M^{\varphi_{\tau_0}} \leq \lambda_M^{\psi}$.

Proof. We have

$$\forall_{\sigma < -\sigma_\varepsilon} |\varphi_{\tau_0}(s)| \leq \sum_n A_n \exp[-\sigma \lambda_n (1+\beta^* \varepsilon)] = \psi(\sigma(1+\beta^* \varepsilon)) ,$$

and hence $M(\sigma; \varphi_{\tau_0}) \leq \psi(\sigma(1+\beta^* \varepsilon))$. As ε is arbitrary, we have $\lambda_M^{\varphi_{\tau_0}} \leq \lambda_M^\psi$.

Remark 5. It is possible to prove this theorem, by using a constant $\beta' > \beta$; in fact we have for $n' = n_\beta$,

$$\begin{aligned} \forall_{\sigma < -\sigma_\varepsilon} |\varphi_{\tau_0}(s) - \sum_{l=1}^{n'-1} P_n(\sigma + i\tau_0) \exp(-s\lambda_n)| &\leq \sum_{n \geq n'} A_n \exp(-\sigma \lambda_n (1+\beta' \varepsilon)) \\ &= \psi_{n'}(\sigma(1+\beta' \varepsilon)) . \end{aligned}$$

For $\lambda^0 > \lambda_{n'-1}$, a constant, we have

$$\exists_{\sigma_\nu (> \sigma_\varepsilon)} \forall_{\sigma < -\sigma_\nu} M(\sigma; \varphi_{\tau_0}) - \exp(-\lambda^0 \sigma) < \psi_{n'}[\sigma(1+\beta' \varepsilon)]$$

from which it follows that $\lambda_M^{\varphi_{\tau_0}} \leq \lambda_M^{\psi_{n'}} = \lambda_M^\psi$.

Theorem 4. If $\sigma_C^\psi = -\infty$, $\beta < \infty$, $L < \infty$, $\lambda_M^\psi < \infty$,

$$\sup \{(\lambda_{n+1} - \lambda_n)/n \mid n \in \mathbb{N} - \{0\}\} = \Lambda < \infty \quad (4)$$

and Φ , defined by (3) in corollary 1, is a non-decreasing function of n for

$n > n_0$, then $\forall_{\tau_0 \in \mathbb{R}} \lambda_M^{\psi} \leq \lambda_M^{\varphi_{\tau_0}}$.

Proof by contradiction. Suppose that the theorem is false. Then

$\exists_{\tau_0 \in \mathbb{R}} \lambda_M^{\varphi_{\tau_0}} < \lambda_M^\psi$. Let $\varepsilon \in]0, \lambda_M^\psi - \lambda_M^{\varphi_{\tau_0}}[$, $\varepsilon_1 \in]0, \varepsilon/\lambda_M^\psi[$ and $\lambda = \frac{\lambda_M^\psi - \varepsilon}{1 - \varepsilon_1}$. Then

we have $\lambda_M^{\varphi_{\tau_0}} < \lambda < \lambda_M^\psi$. If $L < \infty$ and Φ is a non decreasing function of n for $n \geq n_0$, Q.I. Rahman [2] proved that

$$\lambda_M^\psi \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}} ,$$

hence we have

$$\exists_{n'} \forall_{n \geq n'} -\left(\frac{1}{\lambda}\right) \lambda_n \log \lambda_n < \log A_n .$$

From [1] we get the inequality

$$\forall_{\varepsilon > 0} \exists_{n_\varepsilon} \forall_{n \geq n_\varepsilon} A_n < M(\sigma - \lambda'; \varphi_{\tau_0}) \exp\{[(\beta + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|)) + \sigma + \lambda'] \lambda_n\}$$

where λ' is a constant lying in $\mathbb{R}^+ - [0, 1]$. Also we have

$$\forall_{\varepsilon_1 > 0} \exists_{\sigma_{\varepsilon_1} > 0} \forall_{|\sigma| > \sigma_{\varepsilon_1}} (\beta + \varepsilon) \log[6(1 + |\sigma| + |\tau_0|)] < \varepsilon_1 |\sigma| ,$$

and hence

$$\forall_{n \geq n_\varepsilon} \forall_{\sigma < -\sigma_{\varepsilon_1}} A_n < M(\sigma - \lambda'; \varphi_{\tau_0}) \exp\{(\sigma(1 - \varepsilon_1) + \lambda') \lambda_n\}$$

which gives that

$$\forall_{n \geq \max(n', n_\varepsilon)} \forall_{\sigma < -\sigma_{\varepsilon_1}} \exp\left\{-\left(\frac{1}{\lambda}\right) \lambda_n \log \lambda_n - (\sigma(1 - \varepsilon_1) + \lambda') \lambda_n\right\} < M(\sigma - \lambda'; \varphi_{\tau_0}) .$$

Let us consider the mapping

$$\mathbb{R}^+ - \{0\} \ni x \rightarrow \exp\{-\alpha_1 x \log x + \alpha_2 x\}$$

where $\alpha_1 > 0$. This takes the maximum value at $x = \exp\left\{\frac{\alpha_2 - \alpha_1}{\alpha_1}\right\}$. Choosing $\alpha_1 = 1/\lambda$ and $\alpha_2 = -(\sigma(1 - \varepsilon_1) + \lambda')$ we have (where $\varepsilon_1 \in]0, 1[$)

$$\exp\left(\frac{\alpha_2 - \alpha_1}{\alpha_1}\right) = \exp\{-(\sigma(1 - \varepsilon_1) + \lambda')\lambda - 1\}$$

which we denote by x_σ . Then $x_\sigma \rightarrow \infty$ as $\sigma \rightarrow -\infty$. Put $n_\sigma = \inf\{n \in \mathbb{N} / \lambda_n \geq x_\sigma\}$.

Then $\lambda_{n_\sigma} \geq x_\sigma$ and $\lim_{\sigma \rightarrow -\infty} \lambda_{n_\sigma} = \infty$. Hence $\exists_{\sigma'_{\varepsilon_1} \geq \sigma_{\varepsilon_1}} \forall_{\sigma < -\sigma'_{\varepsilon_1}} n_\sigma \geq \max(n', n_\varepsilon)$ and

$\exp(-\alpha_1 \lambda_{n_\sigma} \log \lambda_{n_\sigma} + \alpha_2 \lambda_{n_\sigma}) < M(\sigma - \lambda'; \varphi_{\tau_0})$. Put $\lambda_{n_\sigma} = \theta_\sigma x_\sigma$ (i.e. $\theta_\sigma \geq 1$). Then

$$\theta_\sigma x_\sigma \{-\alpha_1 \log x_\sigma + \alpha_2\} \left[1 - \frac{\alpha_1 \log \theta_\sigma}{(-\alpha_1 \log x_\sigma + \alpha_2)}\right] < \log M(\sigma - \lambda'; \varphi_{\tau_0}) .$$

Since $-\alpha_1 \log x_\sigma + \alpha_2 = \alpha_1$, $\alpha_1 \theta_\sigma x_\sigma (1 - \log \theta_\sigma) < \log M(\sigma - \lambda'; \varphi_{\tau_0})$. Let us suppose that $\Lambda = \sup\{(\lambda_{n+1} - \lambda_n)/n \in \mathbb{N} - \{0\}\}$ is finite. Then $0 \leq \lambda_{n_\sigma} - x_\sigma \leq \Lambda$ and

$$\theta_\sigma \leq 1 + \Lambda x_\sigma^{-1} = 1 + \Lambda \exp\{(\sigma(1 - \varepsilon_1) + \lambda')\lambda + 1\}$$

with $\lim_{\sigma \rightarrow -\infty} \theta_\sigma = 1$. The mapping $\mathbb{R}^+ - \{0\} \ni x \rightarrow x(1 - \log x)$ attains its maximum

at $x = 1$ and hence

$$\forall \varepsilon' \in]0, 1[\quad \exists \eta(\varepsilon') > 0 \quad \forall x \in [1 - \eta, 1 + \eta] \quad 1 \geq x(1 - \log x) > 1 - \varepsilon'.$$

As $\forall \eta > 0 \quad \exists \sigma(\varepsilon') > 0 \quad \forall \sigma < -\sigma_\eta \quad \theta_\sigma \in [1, 1 + \eta]$, we have

$$\forall \sigma < -\sigma_{\eta_{\varepsilon'}} \quad 1 \geq \theta_\sigma (1 - \log \theta_\sigma) > 1 - \varepsilon'$$

and hence for $\sigma < -\max(\sigma_{\varepsilon_1}', \sigma_{\eta_{\varepsilon'}})$, we have

$$\alpha_1 x_\sigma (1 - \varepsilon') < \log M(\sigma - \lambda'; \varphi_{\tau_0}).$$

Hence we get for $\sigma \rightarrow -\infty$

$$\liminf \left(\frac{\log x_\sigma}{-\sigma} \right) \leq \liminf \left\{ \frac{\log \log M(\sigma - \lambda'; \varphi_{\tau_0})}{-\sigma} \right\} = \lambda_M^{\varphi_{\tau_0}},$$

$\liminf \left(\frac{\alpha_2 - \alpha_1}{-\alpha_1 \sigma} \right) = \lambda(1 - \varepsilon') \leq \lambda_M^{\varphi_{\tau_0}}$, where λ satisfies the inequality

$\lambda_M^{\varphi_{\tau_0}} < \lambda < \lambda_M^\psi$. Thus we get a contradiction that $\lambda_M^{\varphi_{\tau_0}} < \lambda_M^{\varphi_{\tau_0}}$ which proves,

under the stated conditions, that it is impossible to find a $\tau_0 \in \mathbb{R}$ such that

$\lambda_M^{\varphi_{\tau_0}} < \lambda_M^\psi$. Hence we have $\lambda_M^\psi \leq \lambda_M^{\varphi_{\tau_0}}$ which completes the proof of the theorem.

From Theorem 3 and Theorem 4, we have

Theorem 5. Under the conditions stated in Theorem 4, we have

$$\forall \tau_0 \in \mathbb{R} \quad \lambda_M^{\varphi_{\tau_0}} = \lambda_M^\psi.$$

As a result of this theorem, we have an expression for $\lambda_M^{\varphi_{\tau_0}}$ in terms of

λ_n and a_n .

Theorem 6. Under the conditions stated in Theorem 4, we have

$$\forall_{\tau_0 \in \mathbb{R}} \lambda_M^{\tau_0} = \liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}} \right\}.$$

Proof. As condition (4) in Theorem 4 implies that $\log \lambda_{n+1} \sim \log \lambda_n$, we have [2, Theorem 1] that

$$\liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}} \right\} \leq \lambda_M^{\psi}.$$

Also with the conditions imposed on Φ and L , we have [2, Theorem 2] that

$$\lambda_M^{\psi} \leq \liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}} \right\}.$$

The two inequalities together with Theorem 5, give

$$\lambda_M^{\tau_0} = \lambda_M^{\psi} = \liminf_{n \rightarrow \infty} \left\{ \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}} \right\}.$$

Let B_{σ_0} be the demi-strip $\{s \in \mathbb{C} / \sigma < -\sigma_0, \tau \in [\tau_1 - \ell, \tau_1 + \ell]\}$ where $\tau_1 \in \mathbb{R}$,

$\ell \in \mathbb{R}^+ - \{0\}$ are constants. Let

$$\forall_{\sigma < -\sigma_0} m(\sigma; B_{\sigma_0}) = \sup_{\tau_0 \in [\tau_1 - \ell, \tau_1 + \ell]} \{m(\sigma; \varphi_{\tau_0})\},$$

$$\forall_{\sigma < -\sigma_0} m(\sigma; \varphi; B_{\sigma_0}) = \sup_{\tau \in [\tau_1 - \ell, \tau_1 + \ell]} \{|\varphi(\sigma + i\tau)|\}$$

and

$$\lambda^{\varphi}(B_{\sigma_0}) = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log \log m(\sigma; \varphi; B_{\sigma_0})}{-\sigma} \right\}.$$

$\lambda^{\varphi}(B_{\sigma_0})$ is the lower order (in the sense of Ritt-Rahman) of φ in the demi-strip. With the help of this let us define the lower order of φ in $\mathbb{C}$ (in the sense of Ritt-Rahman).

Let $\theta > 0$ be a constant, $e_n(\theta) = \left\{ s \in \mathbb{C} / \left| \frac{P_n(s)}{P_n(s+L')} \right| \geq \theta^{m_n} \right\}$ be a set indexed by n on $\mathbb{N} - \{0\}$ and $e(\theta) = \bigcap_n e_n(\theta)$. Let us suppose that

$\exists \theta_0 \in]0, 1[\quad \exists \sigma_0 > 0 \quad \forall (\sigma + i\tau_0) \in e(\theta_0)$, and hence

$$\forall \sigma < -\sigma_0 \quad \forall n \quad \left| \frac{P_n(\sigma + i\tau_0)}{P_n(\sigma + L' + i\tau_0)} \right| \geq \theta_0^{m_n}.$$

Choosing L' such that $\log \theta_0 > \frac{L-L'}{\beta^*}$, ($\beta^* < \infty$), (this implies that $L' > L$), we have

$$\sum_n \theta_0^{-m_n} \exp(-L' \lambda_n) < \sum_n \exp[-\lambda_n (L' + \beta^* \log \theta_0)]$$

(this series is convergent).

Let us denote by $\Omega(L')$ the space of sequences (γ_n) of mappings from $\mathbb{R}$ to $\mathbb{C}$ such that

$$\exists \sigma' > 0 \quad \forall \sigma < -\sigma' \quad \sum_n |\gamma_n(\sigma)| \exp(-L' \lambda_n)$$

is convergent. Let $z = x + iy$ with $(x, y) \in \mathbb{R}^2$. From Theorem A we have for $(\gamma_n) \in \Omega(L')$ and $\operatorname{Re} z = x \geq \sigma + L'$, with $\sigma < -\sigma'$,

$$\sum P_n(\sigma + i\tau_0) \gamma_n(\sigma) \exp(-z \lambda_n) = \lim_{\tau_2} \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} \varphi_{\tau_0}(s) \sum_n \gamma_n(\sigma) \exp(-\lambda_n(z-s)) d\tau, \tau_2 \rightarrow \infty.$$

The series in the left side is convergent, since $\sum_n \gamma_n(\sigma) \exp -\lambda_n(z-s)$ is absolutely convergent for $x \geq \sigma + L'$ and $\sigma < -\sigma'$.

Now let us consider a sequence (γ_n) of mappings from $\mathbb{R}$ to $\mathbb{C}$ such that

$$\forall \sigma < -\sigma_0 \quad \forall n \quad \left| \frac{\gamma_n(\sigma) P_n(\sigma + i\tau_0)}{P_n(\sigma + L' + i\tau_0)} \right| = 1;$$

since $(\sigma + i\tau_0) \in e(\theta_0)$ we have $\left| \frac{1}{\gamma_n(\sigma)} \right| \geq \theta_0^{m_n}$. $\sum |\gamma_n(\sigma)| \exp(-L' \lambda_n)$ is majorised

by the convergent series $\sum \exp(-\lambda_n(L'+\beta^* \log \theta_0))$ and hence it is convergent for $\sigma < -\sigma_0$ which shows that $(\gamma_n) \in \Omega(L')$. Hence for $\sigma < -\sigma_0$,

$$\left| \sum P_n(\sigma+L'+i\tau_0) \exp(-z\lambda_n) \right|^2 \leq m(\sigma; \varphi_{\tau_0}) C(L'+\beta^* \log \theta_0)$$

where $C(L'+\beta^* \log \theta_0)$ is the sum of the series $\sum \exp(-\lambda_n(L'+\beta^* \log \theta_0))$ and $\text{Im } z = y$ is arbitrary in $\mathbb{R}$. Choosing $\text{Re } z = x = \sigma + L'$ and $y = \tau_0$ we have $|\varphi(\sigma+L'+i\tau_0)| \leq m(\sigma; \varphi_{\tau_0}) C(L'+\beta^* \log \theta_0)$.

Let us suppose that $\exists_{\theta_0 \in]0,1]} B_{\sigma_0} \subset E(\theta_0)$. Then we have

$$m(\sigma+L'; \varphi; B_{\sigma_0}) \leq m(\sigma; B_{\sigma_0}) C(L'+\beta^* \log \theta_0).$$

Further for $\varepsilon > 0$

$$\begin{aligned} \forall_{\tau_0 \in [\tau_1-\ell, \tau_1+\ell]} \forall_{\sigma < -\sigma_\varepsilon} m(\sigma; \varphi_{\tau_0}) &\leq \sum A_n \exp[-\sigma \lambda_n + \beta^* \lambda_n \log(1+|\sigma|+|\tau_1|+|\ell|)] \\ &\leq \psi(\sigma(1+\varepsilon\beta^*)); \end{aligned}$$

the last inequality is due to the fact that $\exists_{\sigma_\varepsilon > 0} \forall_{\sigma < -\sigma_\varepsilon} \log(1+|\sigma|+|\tau_1|+|\ell|) < \varepsilon|\sigma|$.

Hence $\forall_{\sigma < -\max(\sigma_0; \sigma_\varepsilon)} m(\sigma+L'; \varphi; B_{\sigma_0}) \leq \psi[\sigma(1+\varepsilon\beta^*)] C(L'+\beta^* \log \theta_0)$ from which it

follows that

$$\lambda^\varphi(B_{\sigma_0}) \leq \lambda_M^\psi(1+\varepsilon\beta^*)$$

and as ε is arbitrary, $\lambda^\varphi(B_{\sigma_0}) \leq \lambda_M^\psi$. On putting (if the above condition is satisfied)

$$\lambda_m^\varphi = \sup\{\lambda^\varphi(B_{\sigma_0})/B_{\sigma_0} = B_{\sigma_0}(\tau_1, \ell) \wedge \tau_1 \in \mathbb{R} \wedge \ell \in \mathbb{R}^+ - \{0\}\}$$

we have $\lambda_m^\varphi \leq \lambda_M^\psi$ and let us define this λ_m^φ as the lower order of φ in $\mathbb{C}$ (in the sense of Ritt-Rahman).

We remark that when $P_n(s)$ is a complex number then our definition coincides with that of Q.I. Rahman for the lower-order of entire functions defined by classical Dirichlet series.

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