

SOME INEQUALITIES FOR KOHN-NIRENBERG PSEUDO-DIFFERENTIAL OPERATORS

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I. INTRODUCTION

An important class of pseudo-differential operators was introduced in the fundamental paper of Kohn and Nirenberg [5]. They consider "symbols" that are complex-valued functions $a(x, \xi)$ defined on the product space $\mathbb{R}^n \times \mathbb{R}^n - \{0\}$ or on $\mathbb{R}^n \times \mathbb{R}^n$, having specific properties, and they associate to these functions two operators $A(x, D)$ and $A(x, D)$, defined by formulas involving Fourier transforms, like

$$A(x, D)u = F^{-1}(a(\infty, \xi)(Fu)(\xi) + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \xi) \hat{u}(\eta) d\eta)$$

$$A(x, D)u = F^{-1}(a(\infty, \xi) \hat{u}(\xi) + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta)$$

where F^{-1} is the inverse Fourier transform, $Fu = \hat{u}$ is the direct Fourier transform, $a'(x, \xi) = a(x, \xi) - a(\infty, \xi)$, and $\hat{a}'(\lambda, \xi)$ is the Fourier transform of $a'(x, \xi)$ with respect to the x -variables.

Afterwards, Kohn and Nirenberg proved various very interesting properties of these operators, most of them having counterparts in previous results of the theory of singular integral operators (Mikhlin [6], Calderon-Zygmund [2], Seeley[7]).

The theory of pseudo-differential operators was the object of much study and research, as in some works by Hörmander [4], Friedrichs [3] and several other authors.

In this manuscript, we took as starting point a very nice presentation of pseudo-differential operators in the Lecture Notes by A.Andreotti [1] where, in a

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clear and detailed manner, a variant to Kohn-Nirenberg's theory is developed.

If we consider Sobolev's spaces $H^s(\mathbb{R}^n)$, where s is a real number, $H^\infty(\mathbb{R}^n) = \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$, $H^{-\infty}(\mathbb{R}^n) = \bigcup_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$, an operator $L: H^\infty \rightarrow H^\infty$ has order ≤ 0 if $\|Lu\|_{H^s} \leq C_s \|u\|_{H^s}$ holds $\forall$ real s ; the order of L equals 0 if any inequality

$$\|Lu\|_{H^s} \leq C_s \|u\|_{H^{s-\varepsilon}}, \quad \forall u \in H^\infty$$

is impossible when ε is > 0 .

Gohberg's Lemma, as given in Andreotti's Lecture Notes, permits to deduce, as a simple corollary, that a pseudo-differential operator associated to a regular symbol $a(x, \xi)$ so that $a(x, t\xi) = a(x, \xi) \quad \forall t > 0, x \in \mathbb{R}^n, \xi \in \mathbb{R}^n - \{0\}$, has exactly the order 0, unless the symbol $a(x, \xi)$ is identical to zero. This Lemma can be remembered as follows. Let $a(x, \xi)$ be a symbol (regular and homogeneous of degree 0 with respect to ξ); let $x_0 \in \mathbb{R}^n, \xi_0 \in \mathbb{R}^n, |\xi_0|_{\mathbb{R}^n} = 1$ be a given point; let s be a nonnegative real number and A a pseudo-differential operator associated to the symbol. Then, there exists a sequence of linear isometric operators $\{P_j\}_1^\infty, H^0 \rightarrow H^s$ so that, for any $u \in S$ -space of C^∞ -rapidly decreasing functions, there is

$$(i) \quad P_j u \rightarrow 0 \text{ in } H^{s-\varepsilon}, \quad \forall \varepsilon > 0 \text{ as } j \rightarrow \infty$$

$$(ii) \quad P_j^{-1} A P_j u \rightarrow a(x_0, \xi_0) u, \quad j \rightarrow \infty, \text{ weakly in } H^0.$$

The main purpose of this paper is to get a similar result (Theorems 3 and 4) for a different class of symbols, such as that introduced in the Appendix of [3], following discussions with I.M.Singer.

We apply the result to get (in Theorem 5) that a pseudo-differential operator has again the order exactly equal to 0, unless $\max_{x \in \mathbb{R}^n} \lim_{|\xi| \rightarrow \infty} |a(x, \xi)| = 0$. As a final remark: the definition of the operators $P_{x_0, \xi_j}^s, Q_{x_0, \xi_j}^s$, which are used in the case of non-regular and non-homogeneous symbols, is inspired by Hörmander ([4], page 158), with $\rho = 1, \delta = 0, \theta = \frac{\rho+\delta}{2} = \frac{1}{2}$; yet the class of symbols we will consider looks quite different from Hörmander's various classes $S_{\rho, \delta}^m$.

II. To simplify matters, as in [5]-Appendix, we consider only bounded symbols: $a(x, \xi) = a(\infty, \xi) + a'(x, \xi)$, where $a'(x, \xi)$ as function of x is convergent to 0 at ∞ , uniformly for $\xi, |\xi| > R$. If $\hat{a}'(\eta, \xi)$ is the Fourier transform of the function a' with respect to x , then

$$(a) \quad |\hat{a}'(\eta, \xi)| \leq k(\eta) \quad \text{where } k(\eta)(1+|\eta|) \text{ is integrable.}$$

(b) There exists a continuous function $\varphi(t) > 0$, defined for $t > 0$, convergent to 0 as $t \rightarrow \infty$, so that

$$|a(x, \xi) - a(x, \tau)| \leq (1 + |\xi - \tau|) \varphi(|\xi|).$$

Further

$$|\hat{a}'(\eta, \xi) - \hat{a}'(\eta, \tau)| \leq (1 + |\xi - \tau|) \varphi(|\xi|) k(\eta)$$

where $k(\eta)$ is the function in (a).

Remark. Kohn-Nirenberg have also another condition: (c) which is not used in the present paper.

Consider now the operator A defined on the S -space of $C^\infty(\mathbb{R}^n)$ — rapidly decreasing functions by the formula

$$(Au)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(x, \eta) \hat{u}(\eta) d\eta.$$

This integral is absolutely convergent, because $a(x, \eta)$ is a bounded symbol and $\hat{u}(\eta) \in S(\mathbb{R}^n)$; consequently, the function $(Au)(x)$ is well-defined for any $x \in \mathbb{R}^n$.

In fact, the function $(Au)(x)$ is continuous and bounded over $\mathbb{R}^n$. Boundedness follows from the estimate

$$|(Au)(x)| \leq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |a(x, \eta)| |\hat{u}(\eta)| d\eta \leq \sup_{\mathbb{R}^n \times \mathbb{R}^n} |a(x, \eta)| \cdot (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\hat{u}(\eta)| d\eta.$$

Continuity follows from the dominated convergence theorem:

$$\text{if } x_p \in \mathbb{R}^n, x_p \rightarrow \bar{x}, \text{ then } e^{ix_p \cdot \eta} a(x_p, \eta) \rightarrow e^{i\bar{x} \cdot \eta} a(\bar{x}, \eta), \quad \forall \eta \in \mathbb{R}^n$$

due to the continuity of $a(x, \xi)$ with respect to x . This last fact follows because

$$a(x, \xi) = a(\infty, \xi) + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \hat{a}'(\eta, \xi) d\eta$$

and

$$|\hat{a}'(\eta, \xi)| \leq k(\eta) \in L^1.$$

Remark also the estimate

$$|e^{ix_p \cdot \eta} a(x_p, \eta) \hat{u}(\eta)| \leq C |\hat{u}(\eta)| \in L^1.$$

The following result is an important tool for the study of the operator A :

Theorem 1. The operator A admits the representation formula:

$$(Au)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta + a(\infty, \xi) \hat{u}(\xi) \right] d\xi$$

for $x \in \mathbb{R}^n$, $u \in S(\mathbb{R}^n)$.

Proof. Remark first the relations:

$$\begin{aligned}
(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(x, \eta) \hat{u}(\eta) d\eta &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} [a'(x, \eta) + a(\infty, \eta)] \hat{u}(\eta) d\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(\infty, \eta) \hat{u}(\eta) d\eta + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a'(x, \eta) \hat{u}(\eta) d\eta.
\end{aligned}$$

Hence it remains to prove only that equality

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a'(x, \eta) \hat{u}(\eta) d\eta = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta \right] d\xi$$

holds, $\forall x \in \mathbb{R}^n$.

Here, the integral $(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta$ is absolutely convergent because the integrand is estimated by $k(\xi - \eta) |\hat{u}(\eta)|$, where $k \in L^1$; also, as a function of ξ , this is an $L^1(\mathbb{R}^n)$ function too. Hence

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta \right] d\xi$$

is absolutely integrable; it equals

$$\begin{aligned}
&(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot (\xi - \eta)} e^{ix \cdot \eta} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{u}(\eta) d\eta \right] d\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot (\xi - \eta)} \hat{a}'(\xi - \eta, \eta) d\xi \right] e^{ix \cdot \eta} \hat{u}(\eta) d\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left[(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \lambda} \hat{a}'(\lambda, \eta) d\lambda \right] e^{ix \cdot \eta} \hat{u}(\eta) d\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a'(x, \eta) \hat{u}(\eta) d\eta.
\end{aligned}$$

Once this formula is proved, it can be deduced that the estimate $\|Au\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}$ for any $u \in S$ is satisfied and that the definition of the operator can be extended to the whole of $L^2(\mathbb{R}^n)$. (Remark that the Hypothesis (a) by itself is enough to prove these results.)

III. Let us define now, for a given point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$, an operator P_{x_0, ξ_0} acting on $S(\mathbb{R}^n)$ by the formula

$$(P_{x_0, \xi_0} u)(x) = |\xi_0|^{\frac{n}{4}} u((x-x_0)|\xi_0|^{\frac{1}{2}}) e^{ix \cdot \xi_0}.$$

One can see that P_{x_0, ξ_0} maps $S(\mathbb{R}^n)$ onto itself; the proof is standard. Let us compute now the Fourier transform

$$(P_{x_0, \xi_0} u)^\wedge(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} |\xi_0|^{\frac{n}{4}} u((x-x_0)|\xi_0|^{\frac{1}{2}}) e^{ix \cdot \xi_0} dx.$$

If $y = (x-x_0)|\xi_0|^{\frac{1}{2}}$, then $dx = |\xi_0|^{-\frac{n}{2}} dy$, hence, we get, as $x = x_0 + |\xi_0|^{\frac{1}{2}} y$, the formula

$$\begin{aligned} (P_{x_0, \xi_0} u)^\wedge(\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(x_0 + |\xi_0|^{\frac{1}{2}} y) \cdot \xi} |\xi_0|^{\frac{n}{4}} u(y) |\xi_0|^{\frac{n}{4}} dy \\ &= e^{-ix_0 \cdot (\xi - \xi_0)} |\xi_0|^{-\frac{n}{4}} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-iy \cdot (\frac{\xi - \xi_0}{|\xi_0|^{\frac{1}{2}}})} u(y) dy \\ &= e^{-ix_0 \cdot (\xi - \xi_0)} |\xi_0|^{-\frac{n}{4}} \hat{u}\left(\frac{\xi - \xi_0}{\sqrt{|\xi_0|}}\right). \end{aligned}$$

At this stage we can also compute, for $u \in S$, the expression for AP_{x_0, ξ_0} given by formula

$$\begin{aligned} (AP_{x_0, \xi_0} u)(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(x, \eta) (P_{x_0, \xi_0} u)^\wedge(\eta) d\eta \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(x, \eta) e^{-ix_0 \cdot (\eta - \xi_0)} |\xi_0|^{-\frac{n}{4}} \hat{u}\left(\frac{\eta - \xi_0}{\sqrt{|\xi_0|}}\right) d\eta. \end{aligned}$$

We shall also consider the operator Q_{x_0, ξ_0} on S , defined through the relation

$$(Q_{x_0, \xi_0} v)(x) = |\xi_0|^{-\frac{n}{4}} v(x_0 + |\xi_0|^{\frac{1}{2}} x) e^{-i(x_0 + |\xi_0|^{\frac{1}{2}} x) \cdot \xi_0}.$$

It is again a linear map of S into itself, as is easily seen and it is also a mapping of $C_\infty(\mathbb{R}^n)$ into itself, where $C_\infty(\mathbb{R}^n)$ is the space of bounded continuous functions on $\mathbb{R}^n$.

Furthermore we have

$$\begin{aligned}
Q_{x_0, \xi_0}(P_{x_0, \xi_0} u)(x) &= |\xi_0|^{-\frac{n}{4}} (P_{x_0, \xi_0} u)(x_0 + |\xi_0|^{-\frac{1}{2}} x) e^{-i(x_0 + |\xi_0|^{-\frac{1}{2}} x) \cdot \xi_0} \\
&= |\xi_0|^{-\frac{n}{4}} \cdot |\xi_0|^{\frac{n}{4}} u(x) e^{i(x_0 + |\xi_0|^{-\frac{1}{2}} x) \cdot \xi_0} e^{-i(x_0 + |\xi_0|^{-\frac{1}{2}} x) \cdot \xi_0} = u(x),
\end{aligned}$$

so that Q_{x_0, ξ_0} is the inverse of P_{x_0, ξ_0} on the linear space S . Let us also compute the expression $(Q_{x_0, \xi_0} A P_{x_0, \xi_0} u)(x)$, $u \in S(1)$, which equals, by the above,

$$\begin{aligned}
&|\xi_0|^{-\frac{n}{4}} e^{-i(x_0 + |\xi_0|^{-\frac{1}{2}} x) \cdot \xi_0} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x_0 + |\xi_0|^{-\frac{1}{2}} x) \cdot \eta} \cdot a(x_0 + |\xi_0|^{-\frac{1}{2}} x, \eta) \cdot \\
&\quad \cdot e^{-i x_0 \cdot (\eta - \xi_0)} |\xi_0|^{-\frac{n}{4}} \hat{u}\left(\frac{\eta - |\xi_0|}{\sqrt{|\xi_0|}}\right) d\eta
\end{aligned}$$

that is

$$\begin{aligned}
&(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x_0 + \frac{x}{\sqrt{|\xi_0|}}) \cdot (\eta - \xi_0)} a(x_0 + \frac{x}{\sqrt{|\xi_0|}}, \eta) e^{-i x_0 \cdot (\eta - \xi_0)} |\xi_0|^{-\frac{n}{2}} \hat{u}\left(\frac{\eta - |\xi_0|}{\sqrt{|\xi_0|}}\right) d\eta \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i \frac{x}{\sqrt{|\xi_0|}} \cdot (\eta - \xi_0)} a(x_0 + x |\xi_0|^{-\frac{1}{2}}, \eta) |\xi_0|^{-\frac{n}{2}} \hat{u}\left(\frac{\eta - |\xi_0|}{\sqrt{|\xi_0|}}\right) d\eta
\end{aligned}$$

(equals, after substitution $\frac{\eta - \xi_0}{\sqrt{|\xi_0|}} = \xi$, $d\eta = |\xi_0|^{\frac{n}{2}} d\xi$):

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i x \cdot \xi} a(x_0 + x |\xi_0|^{-\frac{1}{2}}, \xi \sqrt{|\xi_0|} + \xi_0) \hat{u}(\xi) d\xi,$$

a continuous bounded function on $\mathbb{R}^n$.

We shall also remark the equality

$$\|P_{x_0, \xi_0} u\|_{L^2}^2 = \|Q_{x_0, \xi_0} u\|_{L^2}^2 = \|u\|_{L^2}^2 \quad \forall u \in S.$$

In fact we have

$$\int_{\mathbb{R}^n} |(P_{x_0, \xi_0} u)(x)|^2 dx = \int_{\mathbb{R}^n} |\xi_0|^{\frac{n}{2}} |u((x - x_0) |\xi_0|^{-\frac{1}{2}})|^2 dx = \int_{\mathbb{R}^n} |u(y)|^2 dy$$

(1)

$$\begin{aligned}
u \in S &= P_{x_0, \xi_0} u \in S = A P_{x_0, \xi_0} u \in C_\infty(\mathbb{R}^n) = Q_{x_0, \xi_0} (A P_{x_0, \xi_0} u) \\
&\in C_\infty(\mathbb{R}^n).
\end{aligned}$$

$$\int_{\mathbb{R}^n} |(Q_{x_0, \xi_0} u)(x)|^2 dx = \int_{\mathbb{R}^n} |\xi_0|^{-\frac{n}{2}} |u(x_0 + x |\xi_0|^{-\frac{1}{2}})|^2 dx$$

(equals after substitution $x_0 + x |\xi_0|^{-\frac{1}{2}} = y$):

$$\int_{\mathbb{R}^n} |u(y)|^2 dy.$$

IV. We are able by now to give the following

Theorem 2. Let us assume the symbol $a(x, \xi)$ satisfies (a) and (b) with $\sqrt{t} \varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, and also that $\lim_{p \rightarrow \infty} a(x_0, \xi_p) = C_0 \in \mathbb{C}$ exists for an $x_0 \in \mathbb{R}^n$ and for a sequence $\xi_p \in \mathbb{R}^n$ so that $|\xi_p| \rightarrow \infty$. Then the relation

$$\lim_{p \rightarrow \infty} (Q_{x_0, \xi_p}^{AP} u)(x) = C_0 u(x), \quad \forall u \in S$$

holds, for any $x \in \mathbb{R}^n$ and uniformly on bounded sets in $\mathbb{R}^n$.

Proof. Using the obvious equality $C_0 u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} C_0 e^{ix \cdot \eta} \hat{u}(\eta) d\eta$, we have in fact the relation

$$(Q_{x_0, \xi_p}^{AP} u)(x) - C_0 u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} [a(x_0 + x |\xi_p|^{-\frac{1}{2}}, \eta \sqrt{|\xi_p| + \xi_p}) - C_0] \hat{u}(\eta) d\eta.$$

We write now the decomposition

$$a(x_0 + \frac{x}{\sqrt{|\xi_p|}}, \eta \sqrt{|\xi_p| + \xi_p}) - C_0 = a(x_0 + \frac{x}{\sqrt{|\xi_p|}}, \eta \sqrt{|\xi_p|} + \xi_p) - a(x_0, \xi_p) + a(x_0, \xi_p) - C_0.$$

Then it is

$$|(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} [a(x_0, \xi_p) - C_0] \hat{u}(\eta) d\eta| \leq (2\pi)^{-\frac{n}{2}} |a(x_0, \xi_p) - C_0| \int_{\mathbb{R}^n} |\hat{u}(\eta)| d\eta$$

$\rightarrow 0$ as $p \rightarrow \infty$, uniformly for $x \in \mathbb{R}^n$.

Afterwards, we write also the equality

$$\begin{aligned} & a(x_0 + \frac{x}{\sqrt{|\xi_p|}}, \eta \sqrt{|\xi_p|} + \xi_p) - a(x_0, \xi_p) \\ &= a(x_0 + \frac{x}{\sqrt{|\xi_p|}}, \eta \sqrt{|\xi_p|} + \xi_p) - a(x_0, \eta \sqrt{|\xi_p|} + \xi_p) + a(x_0, \eta \sqrt{|\xi_p|} + \xi_p) - a(x_0, \xi_p). \end{aligned}$$

Let us remark that, from the equality

$$a'(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \hat{a}'(\eta, \xi) d\eta,$$

and because $(1+|\eta|)k(\eta) \in L^1$, we can deduce that partial derivatives $\frac{\partial a'(x, \xi)}{\partial x_j}$ are continuous and bounded on $\mathbb{R}^n$; precisely we have $\frac{\partial a'(x, \xi)}{\partial x_j} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \eta} (i\eta_j) \hat{a}'(\eta, \xi) d\eta$ and the estimate

$$\left| \frac{\partial a'(x, \xi)}{\partial x_j} \right| \leq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\eta| k(\eta) d\eta, \quad \forall x \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R}^n.$$

Also, using equalities:

$$a(x, \xi) = a'(x, \xi) + a(\infty, \xi), \quad a(y, \xi) = a'(y, \xi) + a(\infty, \xi), \quad x, y, \xi \in \mathbb{R}^n,$$

we get:

$$a(x, \xi) - a(y, \xi) = a'(x, \xi) - a'(y, \xi) = \sum_{j=1}^n (x_j - y_j) \frac{\partial a'(x, \xi)}{\partial x_j} \Big|_Z,$$

where $Z \in [x, y]$ (Z can depend on ξ , but this is not important here). Hence, estimating, we see that

$$|a(x, \xi) - a(y, \xi)| = |a'(x, \xi) - a'(y, \xi)| \leq C|x - y|,$$

where C is an absolute constant.

Applying this result to our particular case we get

$$|a(x_0 + \frac{x}{\sqrt{|\xi_p|}}, \eta\sqrt{|\xi_p|} + \xi_p) - a(x_0, \eta\sqrt{|\xi_p|} + \xi_p)| \leq C \frac{|x|}{\sqrt{|\xi_p|}}.$$

The corresponding integral is then estimated by $(2\pi)^{-n/2} C \frac{|x|}{\sqrt{|\xi_p|}} \int_{\mathbb{R}^n} |\hat{u}(\eta)| d\eta$; this expression $\rightarrow 0$ as $p \rightarrow \infty$, uniformly for x in bounded sets of $\mathbb{R}^n$.

Finally, we have by the first condition in (b), the inequality

$$|a(x_0, \eta\sqrt{|\xi_p|} + \xi_p) - a(x_0, \xi_p)| \leq (1 + |\eta| |\xi_p|^{\frac{1}{2}}) \varphi(|\xi_p|)$$

and the corresponding integral is then estimated by

$$\begin{aligned} & (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (1 + |\eta| \sqrt{|\xi_p|}) \varphi(|\xi_p|) |\hat{u}(\eta)| d\eta \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(|\xi_p|) |\hat{u}(\eta)| d\eta + (2\pi)^{-\frac{n}{2}} \sqrt{|\xi_p|} \varphi(|\xi_p|) \int_{\mathbb{R}^n} |\eta| |\hat{u}(\eta)| d\eta; \end{aligned}$$

as $\varphi(|\xi_p|) \rightarrow 0$ for $p \rightarrow \infty$, and also $\sqrt{|\xi_p|} \varphi(|\xi_p|) \rightarrow 0$ for $p \rightarrow \infty$, this last expression tends to 0, uniformly for $x \in \mathbb{R}^n$.

The above arguments conclude the proof of Theorem 2.

V. Let us remember the definition of Sobolev norms $\|\cdot\|_s$, on $S(\mathbb{R}^n)$, s being any real, which are given by the expression

$$\|u\|_S = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad \forall u \in S^{(1)}.$$

We shall prove first quite a simple result in form of

Proposition 1. If $x_0 \in \mathbb{R}^n$ and $\xi_j \in \mathbb{R}^n$ is convergent to ∞ , then $\lim_{j \rightarrow \infty} P_{x_0, \xi_j} u = 0$ in the norm of $H_{-\varepsilon}$, $\forall \varepsilon > 0$ and $\forall u \in S$.

Proof. We have already seen that the formula

$$(P_{x_0, \xi_j} u) \wedge (\xi) = e^{-ix_0 \cdot (\xi - \xi_j)} |\xi_j|^{-\frac{n}{2}} \hat{u}\left(\frac{\xi - \xi_j}{|\xi_j|}\right)$$

is satisfied so that

$$\begin{aligned} & \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\varepsilon} |(P_{x_0, \xi_j} u) \wedge (\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\varepsilon} |\xi_j|^{-\frac{n}{2}} \left| \hat{u}\left(\frac{\xi - \xi_j}{|\xi_j|}\right) \right|^2 d\xi = \end{aligned}$$

(after substitution $\frac{\xi - \xi_j}{|\xi_j|} = \eta$):

$$\int_{\mathbb{R}^n} (1+|\eta\sqrt{|\xi_j|} + \xi_j|^2)^{-\varepsilon} |\hat{u}(\eta)|^2 d\eta.$$

Remark now the simple estimate

$$|\xi_j + \eta\sqrt{|\xi_j|}| \geq |\xi_j| - |\eta|\sqrt{|\xi_j|} = \sqrt{|\xi_j|} (\sqrt{|\xi_j|} - |\eta|).$$

If we fix $\eta \in \mathbb{R}^n$, and let $j \rightarrow \infty$, then $|\xi_j + \eta\sqrt{|\xi_j|}| \rightarrow \infty$ too, so that

$$\frac{1}{(1+|\eta\sqrt{|\xi_j|} + \xi_j|^2)^{\varepsilon}} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad \forall \varepsilon > 0, \quad \forall \eta \in \mathbb{R}^n. \quad \text{Applying the dominated}$$

convergence theorem, we obtain henceforth that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} (1+|\eta\sqrt{|\xi_j|} + \xi_j|^2)^{-\varepsilon} |\hat{u}(\eta)|^2 d\eta = 0, \quad \forall u \in S(\mathbb{R}^n),$$

which ends the proof.

Consider now the map I_s , which is defined for any real s ; if $u \in S(\mathbb{R}^n)$, then $\hat{u}(\xi) \in S(\mathbb{R}^n)$ and $(1+|\xi|^2)^{s/2} \hat{u}(\xi) \in S$ too; then, let us define $(I_s u)(x) = F^{-1}[(1+|\xi|^2)^{s/2} \hat{u}(\xi)] = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1+|\xi|^2)^{s/2} \hat{u}(\xi) d\xi \in S(\mathbb{R}^n)$.

Then let take into consideration the operator $I_{-s} P_{x_0, \xi_j}$; remark that, $\forall u \in S$, we have

- (1) The Sobolev space H_s can be defined as the completion of the space S with respect to the norm $\|\cdot\|_s$.

$$\begin{aligned} \|I_{-s}^P P_{x_0, \xi_j} u\|_{H_{s-\varepsilon}} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-\varepsilon} |(I_{-s}^P P_{x_0, \xi_j} u) \wedge (\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-\varepsilon} (1+|\xi|^2)^{-s} |(P_{x_0, \xi_j} u) \wedge (\xi)|^2 d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\varepsilon} |(P_{x_0, \xi_j} u) \wedge (\xi)|^2 d\xi, \end{aligned}$$

which $\rightarrow 0$ as $j \rightarrow \infty$ if we keep the hypothesis of Proposition 1 ($x_0 \in \mathbb{R}^n$, $\xi_j \rightarrow \infty$).

Let us denote by P_{x_0, ξ_j}^S the linear operator $S \rightarrow S$ given by $P_{x_0, \xi_j}^S =$

$I_{-s}^P P_{x_0, \xi_j}$, and let us remark the equality

$$\|P_{x_0, \xi_j}^S u\|_{H_S} = \|P_{x_0, \xi_j} u\|_{L^2} = \|u\|_{L^2}, \quad \forall u \in S.$$

We can also consider the operator $Q_{x_0, \xi_j} I_S$, which maps $S(\mathbb{R}^n)$ into itself. For any u of $S(\mathbb{R}^n)$, $\|Q_{x_0, \xi_j} I_S u\|_{H_0} = \|I_S u\|_{H_0} = \|u\|_{H_S}$. Denoting this operator by Q_{x_0, ξ_j}^S we see that

$$P_{x_0, \xi_j}^S Q_{x_0, \xi_j}^S u = I_{-s}^P P_{x_0, \xi_j} Q_{x_0, \xi_j} I_S u = I_{-s}^P I_S u = u, \quad \forall u \in S.$$

Further, we will be interested to consider the sequence of operators $Q_{x_0, \xi_j}^S A P_{x_0, \xi_j}^S$; as A does not map $S(\mathbb{R}^n)$ into itself, we must add a few supplementary remarks, in order to give a meaning to these operators. Remember that the space $S'(\mathbb{R}^n)$ -dual to $S(\mathbb{R}^n)$ - is the space of temperate distributions on $\mathbb{R}^n$; for any real σ , the Hilbertian subspace H_σ is also defined as consisting of those $T \in S'(\mathbb{R}^n)$ which have Fourier transform $\hat{T}$ being a locally integrable function of ξ , so that

$\int_{\mathbb{R}^n} (1+|\xi|^2)^\sigma |\hat{T}(\xi)|^2 d\xi < \infty$. Then the operator I_S extends as an operator $S' \rightarrow S'$: let us put $(I_S T)(\varphi) = T(I_S \varphi)$, $\forall u \in S$, $T \in S'$ and let us remark that $\varphi \rightarrow I_S$ is a linear continuous map on $S(\mathbb{R}^n)$.

The operator I_S is important essentially because it is an isometric linear map of $H_\sigma(\mathbb{R}^n)$ onto $H_{\sigma-s}(\mathbb{R}^n)$, and this for any real σ ; in particular, I_{-s} is a linear isometry of H_0 onto H_s .

Consider again the operator $I_{-s}^P P_{x_0, \xi_j} = P_{x_0, \xi_j}^S$; due to the relation $\|P_{x_0, \xi_j}^S u\|_{H_S} = \|u\|_{H_0}$, $\forall u \in S$, it extends by continuity from S to $L^2(\mathbb{R}^n)$, as a map $H_0 \rightarrow H_S$, which is linear and isometric.

The operator A , when applied to $P_{x_0, \xi_j}^S u$, $u \in S$, gives a bounded continuous

function; if we assume that the function $k(\eta)$ in section II verifies the supplementary condition

$$(1+|\eta|^2)^{\frac{|s|}{2}} \cdot k(\eta) \in L^1,$$

then A extends to a linear continuous map: $H_s \rightarrow H_s$. Hence $A(P_{x_0, \xi_j}^S u) \in H_s$, $\forall u \in S$. The operator $Q_{x_0, \xi_j}^S A(P_{x_0, \xi_j}^S) = Q_{x_0, \xi_j}^S I_S A P_{x_0, \xi_j}^S$ is well defined because Q_{x_0, ξ_j}^S also extends as a linear isometric map $H_s \rightarrow H_0$, so that, altogether, the operator $Q^S A P^S$ can be considered as a linear map $H_0 \rightarrow H_0$.

We shall now look for an explicit representation of $Q^S A P^S u$, for u belonging to S .

We shall use Theorem 1; as we remarked that $A v$ is continuous and bounded on $\mathbb{R}^n$, it results that $A v$ defines a temperate distribution for $v \in S$; hence, its Fourier transform (in $S'(\mathbb{R}^n)$ sense) equals

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) \hat{v}(\eta) d\eta + a(\infty, \xi) \hat{v}(\xi).$$

In our case $v = I_{-s} P_{x_0, \xi_j} u$; the Fourier transform $\hat{v}(\xi)$ is given by

$$\hat{v}(\xi) = (1+|\xi|^2)^{-s/2} (P_{x_0, \xi_j} u) \wedge (\xi) = (1+|\xi|^2)^{-s/2} |\xi_j|^{-\frac{n}{4}} \hat{u}\left(\frac{\xi - \xi_j}{|\xi_j|}\right) \cdot e^{-ix_0 \cdot (\xi - \xi_j)}.$$

Consequently we have the formula

$$\begin{aligned} & (A P_{x_0, \xi_j}^S u) \wedge (\xi) \\ &= a(\infty, \xi) (1+|\xi|^2)^{-\frac{s}{2}} |\xi_j|^{-\frac{n}{4}} \hat{u}\left(\frac{\xi - \xi_j}{|\xi_j|}\right) e^{-ix_0 \cdot (\xi - \xi_j)} \\ &+ (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) (1+|\eta|^2)^{-\frac{s}{2}} |\xi_j|^{-\frac{n}{4}} \hat{u}\left(\frac{\eta - \xi_j}{|\xi_j|}\right) e^{-ix_0 \cdot (\eta - \xi_j)} d\eta. \end{aligned}$$

Then we have to consider the expression

$$I_S A P_{x_0, \xi_j}^S u;$$

we see that

$$\begin{aligned} & (I_S (A P_{x_0, \xi_j}^S u)) \wedge (\xi) = (1+|\xi|^2)^{\frac{s}{2}} (A P_{x_0, \xi_j}^S u) \wedge (\xi) \\ &= a(\infty, \xi) |\xi_j|^{-\frac{n}{4}} e^{-ix_0 \cdot (\xi - \xi_j)} \hat{u}\left(\frac{\xi - \xi_j}{|\xi_j|}\right) \\ &+ (2\pi)^{-\frac{n}{2}} (1+|\xi|^2)^{\frac{s}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi - \eta, \eta) (1+|\eta|^2)^{-\frac{s}{2}} |\xi_j|^{-\frac{n}{4}} \hat{u}\left(\frac{\eta - \xi_j}{|\xi_j|}\right) e^{-ix_0 \cdot (\eta - \xi_j)} d\eta. \end{aligned}$$

Remark now that the formula

$$\begin{aligned} (Q_{x_0, \xi_j} w) \wedge (\xi) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} |\xi_j|^{-\frac{n}{4}} w\left(\frac{x}{\sqrt{|\xi_j|}} + x_0\right) e^{-i(x_0 + \frac{x}{\sqrt{|\xi_j|}}) \cdot \xi_j} dx \\ &= |\xi_j|^{\frac{n}{4}} e^{ix_0 \cdot \xi \sqrt{|\xi_j|}} \widehat{w}(\xi \sqrt{|\xi_j|} + \xi_j), \end{aligned}$$

$\forall w \in S$ is satisfied. If we denote by R_{x_0, ξ_j} the operator on $S(\mathbb{R}_\xi^n)$ given by

$$(R_{x_0, \xi_j} Z)(\xi) = |\xi_j|^{\frac{n}{4}} e^{ix_0 \cdot \xi \sqrt{|\xi_j|}} Z(\xi \sqrt{|\xi_j|} + \xi_j),$$

we see that it maps $S \rightarrow S$; also R_{x_0, ξ_j} can be defined by the same formula directly on $L^2(\mathbb{R}^n)$; we see also that, $\forall Z \in L^2$, it follows that

$$\|R_{x_0, \xi_j} Z\|_{L^2}^2 = \int_{\mathbb{R}^n} |\xi_j|^{\frac{n}{2}} |Z(\xi \sqrt{|\xi_j|} + \xi_j)|^2 d\xi = \int_{\mathbb{R}^n} |Z(\eta)|^2 d\eta.$$

Hence we have

$$(Q_{x_0, \xi_j} w) \wedge (\xi) = (R_{x_0, \xi_j} \widehat{w})(\xi), \quad \forall w \in S. \quad (*)$$

Also the same formulas used to define Q_{x_0, ξ_j} and R_{x_0, ξ_j} on S permit to define these operators on $L^2(\mathbb{R}^n)$, where they are linear isometric; it follows that (*) is true for any $w \in L^2$ too.

Continuing now our computation we arrive at the expression: (for any $u \in S$)

$$\begin{aligned} (Q_{x_0, \xi_j} (I_{S^{AP^S}}^{x_0, \xi_j} u) \wedge (\xi)) &= (R_{x_0, \xi_j} (I_{S^{AP^S}}^{x_0, \xi_j} u) \wedge (\xi)) \\ &= R_{x_0, \xi_j} [a(\cdot, \xi) |\xi_j|^{-\frac{n}{4}} e^{-ix_0 \cdot (\xi - \xi_j)} \widehat{u}\left(\frac{\xi - \xi_j}{\sqrt{|\xi_j|}}\right) \\ &\quad + (2\pi)^{-\frac{n}{2}} (1 + |\xi|^2)^{\frac{s}{2}} \int_{\mathbb{R}^n} \widehat{a}'(\xi - \eta, \eta) (1 + |\eta|^2)^{-\frac{s}{2}} |\xi_j|^{-\frac{n}{4}} \widehat{u}\left(\frac{\eta - \xi_j}{\sqrt{|\xi_j|}}\right) e^{-ix_0 \cdot (\eta - \xi_j)} d\eta] \\ &= |\xi_j|^{\frac{n}{4}} e^{ix_0 \cdot \xi \sqrt{|\xi_j|}} e^{-ix_0 \cdot \xi \sqrt{|\xi_j|}} \widehat{u}(\xi) a(\cdot, \xi \sqrt{|\xi_j|} + \xi_j) |\xi_j|^{-\frac{n}{4}} \\ &\quad + (2\pi)^{-\frac{n}{2}} |\xi_j|^{\frac{n}{4}} e^{ix_0 \cdot \xi \sqrt{|\xi_j|}} (1 + |\xi \sqrt{|\xi_j|} + \xi_j|^2)^{\frac{s}{2}} \\ &\quad \int_{\mathbb{R}^n} \widehat{a}'(\xi \sqrt{|\xi_j|} + \xi_j - \eta, \eta) \cdot (1 + |\eta|^2)^{-\frac{s}{2}} |\xi_j|^{-\frac{n}{4}} \widehat{u}\left(\frac{\eta - \xi_j}{\sqrt{|\xi_j|}}\right) e^{-ix_0 \cdot (\eta - \xi_j)} d\eta \end{aligned}$$

$$\begin{aligned}
&= a(\infty, \xi\sqrt{|\xi_j|} + \xi_j) \hat{u}(\xi) + e^{ix_0 \cdot \xi\sqrt{|\xi_j|}} (1 + |\xi\sqrt{|\xi_j|} + \xi_j|^2)^{s/2} \\
&\quad (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\xi\sqrt{|\xi_j|} + \xi_j - \eta, \eta) (1 + |\eta|^2)^{-\frac{s}{2}} e^{-ix_0 \cdot (\eta - \xi_j)} \hat{u}\left(\frac{\eta - \xi_j}{\sqrt{|\xi_j|}}\right) d\eta \\
&= a(\infty, \xi\sqrt{|\xi_j|} + \xi_j) \hat{u}(\xi) + e^{ix_0 \cdot \xi\sqrt{|\xi_j|}} (1 + |\xi\sqrt{|\xi_j|} + \xi_j|^2)^{\frac{s}{2}} \cdot \\
&\quad (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi\sqrt{|\xi_j|} + \xi_j - \sigma) (1 + |\xi\sqrt{|\xi_j|} + \xi_j - \sigma|^2)^{-\frac{s}{2}} e^{-ix_0 \cdot (\xi\sqrt{|\xi_j|} - \sigma)} \hat{u}\left(\xi - \frac{\sigma}{\sqrt{|\xi_j|}}\right) d\sigma \\
&= a(\infty, \xi\sqrt{|\xi_j|} + \xi_j) \hat{u}(\xi) \\
&\quad + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi\sqrt{|\xi_j|} + \xi_j - \sigma) \left(\frac{1 + |\xi\sqrt{|\xi_j|} + \xi_j|^2}{1 + |\xi\sqrt{|\xi_j|} + \xi_j - \sigma|^2} \right)^{\frac{s}{2}} e^{ix_0 \cdot \sigma} \hat{u}\left(\xi - \frac{\sigma}{\sqrt{|\xi_j|}}\right) d\sigma \\
&= (Q_{x_0, \xi_j}^s \text{Ap}_{x_0, \xi_j}^s u) \wedge (\xi).
\end{aligned}$$

Now, we can remark that the last obtained integral $\forall \xi \in \mathbb{R}^n$ is an absolutely convergent one. In fact, we have

$$|\hat{a}'(\sigma, \xi\sqrt{|\xi_j|} + \xi_j - \sigma)| \leq k(\sigma), \quad \forall \sigma \in \mathbb{R}^n.$$

Also, using Peetre's inequality we get

$$\left(\frac{1 + |\xi\sqrt{|\xi_j|} + \xi_j|^2}{1 + |\xi\sqrt{|\xi_j|} + \xi_j - \sigma|^2} \right)^{\frac{s}{2}} \leq 2^{\frac{|s|}{2}} (1 + |\sigma|^2)^{-\frac{|s|}{2}},$$

and finally $\sup_{\sigma \in \mathbb{R}^n} |\hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_j|}})| = M < \infty$, because $\hat{u} \in S$. Hence, the term under the integral sign is estimated by

$$k(\sigma) 2^{\frac{|s|}{2}} (1 + |\sigma|^2)^{-\frac{|s|}{2}} M \in L^1,$$

because we have assumed that $(1 + |\sigma|^2)^{-\frac{|s|}{2}} k(\sigma) \in L^1$ for this part of our work. Hence, the function $(Q_{x_0, \xi_j}^s \text{Ap}_{x_0, \xi_j}^s u) \wedge (\xi)$ is well-defined and finite, for any $\xi \in \mathbb{R}^n$.

Let us choose now (as in Th.2) x_0 and $\xi_j \in \mathbb{R}^n$ in such a way that $|\xi_j| \rightarrow \infty$ and $a(x_0, \xi_j) \rightarrow C_0 \in \mathbb{C}$ as $j \rightarrow \infty$; now, because $a'(x_0, \xi_j) = a(x_0, \xi_j) -$

$a(\infty, \xi_{j_k})$, we can, as $a(\infty, \xi)$ is bounded on $\mathbb{R}^n$, consider a subsequence $(\xi_{j_k})_{k=1}^\infty \rightarrow \infty$, so that $a(\infty, \xi_{j_k}) \rightarrow b_0 \in \mathbb{C}$. Then $a(x_0, \xi_{j_k}) \rightarrow c_0$ too, so that $a'(x_0, \xi_{j_k}) \rightarrow c_0 - b_0 = d_0 \in \mathbb{C}$. Let us also remark that for any ξ of $\mathbb{R}^n$, $\lim_{k \rightarrow \infty} a(\infty, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}) \hat{u}(\xi) = b_0 \hat{u}(\xi)$, provided that $\sqrt{t} \varphi(t) \rightarrow 0$ as $t \rightarrow \infty$. We have in fact — letting $x \rightarrow \infty$ in (b) of section II —

$$|a(\infty, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}) - a(\infty, \xi_{j_k})| \leq (1 + |\xi| \sqrt{|\xi_{j_k}|}) \varphi(|\xi_{j_k}|).$$

So:

$$\begin{aligned} |a(\infty, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}) - b_0| &\leq |a(\infty, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}) - a(\infty, \xi_{j_k})| + |a(\infty, \xi_{j_k}) - b_0| \\ &\leq (1 + |\xi| \sqrt{|\xi_{j_k}|}) \varphi(|\xi_{j_k}|) + |a(\infty, \xi_{j_k}) - b_0| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad \forall \xi \in \mathbb{R}^n, \end{aligned}$$

if we assume $\sqrt{t} \varphi(t) \rightarrow 0$ as $t \rightarrow \infty$ (as we did in Theorem 2).

Let us now consider the expression

$$\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2}$$

appearing in the above given integral. It can be written in the form:

$$\frac{1 + \left| \frac{\xi}{\sqrt{|\xi_{j_k}|}} + \frac{\xi_{j_k}}{|\xi_{j_k}|} \right|^2 |\xi_{j_k}|^2}{1 + \left| \frac{\xi}{\sqrt{|\xi_{j_k}|}} + \frac{\xi_{j_k}}{|\xi_{j_k}|} - \frac{\sigma}{|\xi_{j_k}|} \right|^2 |\xi_{j_k}|^2} = \frac{\frac{1}{|\xi_{j_k}|^2} + \left| \frac{\xi}{\sqrt{|\xi_{j_k}|}} + \frac{\xi_{j_k}}{|\xi_{j_k}|} \right|^2}{\frac{1}{|\xi_{j_k}|^2} + \left| \frac{\xi}{\sqrt{|\xi_{j_k}|}} + \frac{\xi_{j_k}}{|\xi_{j_k}|} - \frac{\sigma}{|\xi_{j_k}|} \right|^2}.$$

Remark that $\frac{\xi_{j_k}}{|\xi_{j_k}|}$ is on the unit sphere in $\mathbb{R}^n$; an infinite subsequence can be chosen⁽¹⁾, so that $\frac{\xi_{j_k}}{|\xi_{j_k}|} \rightarrow \xi_0$ on the unit sphere of $\mathbb{R}^n$. As $|\xi_{j_k}| \rightarrow \infty$ with k , the whole expression is convergent to $\frac{|\xi_0|^2}{|\xi_0|^2} = 1$, and this for any $\sigma \in \mathbb{R}^n$.

We wish now to find the limit of the expression:

(1) for which we keep the same notation $(\xi_{j_k})_{k=1}^\infty$.

$$I_k(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma) \left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} e^{ix_0 \cdot \sigma} \\ \cdot \hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}) d\sigma \quad \text{as } k \rightarrow \infty.$$

We can write the obvious decomposition

$$I_k(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} [\hat{a}'(\sigma, \xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma) - \hat{a}'(\sigma, \xi_{j_k})] \\ \cdot \left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} e^{ix_0 \cdot \sigma} \hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}) d\sigma \\ + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi_{j_k}) \left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} e^{ix_0 \cdot \sigma} \hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}) d\sigma \\ = J_k(\xi) + L_k(\xi).$$

We can also write the obvious relation

$$L_k(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi_{j_k}) e^{ix_0 \cdot \sigma} \hat{u}(\xi) d\sigma \\ + (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{a}'(\sigma, \xi_{j_k}) e^{ix_0 \cdot \sigma} \left[\left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} \hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}) - \hat{u}(\xi) \right] d\sigma \\ = M_k(\xi) + N_k(\xi).$$

Actually, $M_k(\xi) = \hat{u}(\xi) \cdot a'(x_0, \xi_{j_k})$, so that

$$\lim_{k \rightarrow \infty} M_k(\xi) = (c_0 - b_0) \hat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

We will see that $\lim_{k \rightarrow \infty} J_k(\xi) = 0$. We estimate and obtain, without any difficulty, the relation:

$$|J_k(\xi)| \leq (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} (1 + |\xi \sqrt{|\xi_{j_k}|} - \sigma|)^{-\sigma} \phi(|\xi_{j_k}|) k(\sigma) \cdot 2^{\frac{|s|}{2}} (1 + |\sigma|^2)^{\frac{|s|}{2}} \left| \hat{u}\left(\frac{\xi \sqrt{|\xi_{j_k}|} - \sigma}{\sqrt{|\xi_{j_k}|}}\right) \right| d\sigma.$$

Denoting $\frac{\xi \sqrt{|\xi_{j_k}|} - \sigma}{\sqrt{|\xi_{j_k}|}} = \mu$, we get

$$(1 + |\xi \sqrt{|\xi_{j_k}|} - \sigma|) = 1 + |\mu| \sqrt{|\xi_{j_k}|} \leq \sqrt{|\xi_{j_k}|} (1 + |\mu|)$$

for k big enough as $|\xi_{j_k}| \rightarrow \infty$. Hence we have

$$\begin{aligned} (1 + |\xi \sqrt{|\xi_{j_k}|} - \sigma|) \left| \hat{u}\left(\frac{\xi \sqrt{|\xi_{j_k}|} - \sigma}{\sqrt{|\xi_{j_k}|}}\right) \right| &= (1 + |\mu| \sqrt{|\xi_{j_k}|}) |\hat{u}(\mu)| \\ &\leq \sqrt{|\xi_{j_k}|} (1 + |\mu|) |\hat{u}(\mu)| \leq M \sqrt{|\xi_{j_k}|}, \end{aligned}$$

where $M = \sup_{\eta \in \mathbb{R}^n} (1 + |\eta|) |\hat{u}(\eta)| < \infty$. Consequently it will follow that

$$|J_k(\xi)| \leq C_s \sqrt{|\xi_{j_k}|} \phi(|\xi_{j_k}|) \int_{\mathbb{R}^n} k(\sigma) (1 + |\sigma|^2)^{\frac{|s|}{2}} d\sigma \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

(Again, remember that we assume $\sqrt{t} \phi(t) \rightarrow 0$ as $t \rightarrow \infty$) Hence $\lim_{k \rightarrow \infty} J_k(\xi) = 0$.

Let us prove that for every ξ of $\mathbb{R}^n$, $\lim_{k \rightarrow \infty} N_k(\xi) = 0$. We shall apply Lebesgue's dominated convergence theorem; the expression under the integral sign is estimated by:

$$|\hat{a}'(\sigma, \xi_{j_k})| 2^{\frac{|s|}{2}} (1 + |\sigma|^2)^{\frac{|s|}{2}} \left| \hat{u}\left(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}\right) \right| |\hat{u}(\xi)| \leq 2MC_s k(\sigma) (1 + |\sigma|^2)^{\frac{|s|}{2}},$$

where $M = \sup_{\tau \in \mathbb{R}^n} |\hat{u}(\tau)|$, so by an $L^1(\mathbb{R}^n)$ -function. Also, when $\sigma \in \mathbb{R}^n$ is fixed (and $\xi \in \mathbb{R}^n$ too!) we have the inequality

$$\begin{aligned} &\left| \hat{a}'(\sigma, \xi_{j_k}) e^{ix_0 \cdot \sigma} \left[\left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} \hat{u}\left(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}\right) - \hat{u}(\sigma) \right] \right| \\ &\leq k(\sigma) \left\| \left[\left(\frac{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2}{1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2} \right)^{\frac{s}{2}} \hat{u}\left(\xi - \frac{\sigma}{\sqrt{|\xi_{j_k}|}}\right) - \hat{u}(\xi) \right] \right\| \end{aligned}$$

which $\rightarrow 0$ as $k \rightarrow \infty$ for all σ where $k(\sigma) < \infty$ (hence almost-everywhere),

and for all $\xi \in \mathbb{R}^n$, this because $(1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k}|^2)^{\frac{s}{2}} (1 + |\xi \sqrt{|\xi_{j_k}|} + \xi_{j_k} - \sigma|^2)^{-\frac{s}{2}} \rightarrow 1$

and $\hat{u}(\xi - \frac{\sigma}{\sqrt{|\xi_j|}}) \rightarrow \hat{u}(\xi)$ as $k \rightarrow \infty$, $\forall \sigma \in \mathbb{R}^n$.

Summing together the relations proven above, we get

$$\lim_{k \rightarrow \infty} I_k(\xi) = (C_0 - \text{bo})\hat{u}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

It follows that

$$\lim_{k \rightarrow \infty} (Q_{x_0, \xi_{j_k}}^S AP_{x_0, \xi_{j_k}}^S) \wedge (\xi) = \text{bo} \hat{u}(\xi) + (C_0 - \text{bo})\hat{u}(\xi) = C_0 \hat{u}(\xi),$$

where $C_0 = \lim_{j \rightarrow \infty} a(x_0, \xi_j)$ and $(\xi_{j_k})_{k=1}^\infty$ is a convenient second subsequence of $(\xi_j)_{j=1}^\infty$.

We can summarize all the above arguments in the following:

Theorem 3. Let us have the symbol $a(x, \xi)$ which satisfies (a) and (b) with $\sqrt{t}\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, and $(1+|\eta|^2)^{|s|/2} k(\eta) \in L^1(\mathbb{R}^n)$, and assume that $\lim_{p \rightarrow \infty} a(x_0, \xi_p) = C_0 \in \mathbb{C}$ exists for an $x_0 \in \mathbb{R}^n$ and for a sequence $\xi_p \in \mathbb{R}^n$, $|\xi_p| \rightarrow \infty$. Then, there is a subsequence $(\xi_{p_k})_{k=1}^\infty$ so that

$$\lim_{k \rightarrow \infty} (Q_{x_0, \xi_{p_k}}^S AP_{x_0, \xi_{p_k}}^S u) \wedge (\xi) = C_0 \hat{u}(\xi),$$

$\forall u \in S(\mathbb{R}^n)$ and $\forall \xi \in \mathbb{R}^n$, where

$$Q_{x_0, \xi_j}^S = Q_{x_0, \xi_j} I_S, \quad P_{x_0, \xi_j}^S = I_{-S} P_{x_0, \xi_j}.$$

VI. Here we shall firstly see the following:

Theorem 4. Under the conditions of Theorem 3, for any u of $L^2(\mathbb{R}^n)$ we have

$$\lim_{k \rightarrow \infty} Q_{x_0, \xi_{p_k}}^S AP_{x_0, \xi_{p_k}}^S u = C_0 u$$

in the weak topology of $L^2(\mathbb{R}^n)$.

Proof. Remember that $Q_{x_0, p_k}^S AP_{x_0, p_k}^S \in \mathcal{L}(L^2; L^2)$, so that the sequence

$$(v_k(x))_{k=1}^\infty = (Q_{x_0, \xi_{p_k}}^S AP_{x_0, \xi_{p_k}}^S u)_{k=1}^\infty$$

is in $L^2(\mathbb{R}^n)$. It is also an uniformly bounded sequence in L^2 because of the estimates

$$\|v_k\|_{L^2} = \|Q_{x_0, \xi_{p_k}}^S (AP_{x_0, \xi_{p_k}}^S u)\|_{L^2} = \|AP_{x_0, \xi_{p_k}}^S u\|_{H^S} \leq$$

$$\|A\|_{\mathcal{L}(H^S; H^S)} \|P_{x_0, \xi_{p_k}}^S u\|_{H^S} = \|A\|_{\mathcal{L}(H^S; H^S)} \|u\|_{L^2}.$$

Let us prove first that

$$\lim_{k \rightarrow \infty} Q_{x_0, \xi p_k}^S AP_{x_0, \xi p_k}^S u = C_0 u$$

in the weak topology of $L^2(\mathbb{R}^n)$, $\forall u \in S(\mathbb{R}^n)$; this means: $\forall h(x) \in L^2(\mathbb{R}^n)$, it should be

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} (T_k u)(x) \bar{h}(x) dx = C_0 \int_{\mathbb{R}^n} u(x) \bar{h}(x) dx,$$

where $T_k = Q_{x_0, \xi p_k}^S AP_{x_0, \xi p_k}^S$ or, according to Parseval's equality,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} (T_k u) \wedge (\xi) \bar{\tilde{h}}(\xi) d\xi = C_0 \int_{\mathbb{R}^n} \hat{u}(\xi) \bar{\tilde{h}}(\xi) d\xi, \quad \forall h \in L^2(\mathbb{R}^n).$$

Now, $\lim_{k \rightarrow \infty} (T_k u) \wedge (\xi) = C_0 \hat{u}(\xi) \quad \forall \xi \in \mathbb{R}^n$ (Theorem 3). Moreover, $\|(T_k u) \wedge\|_{L^2} = \|T_k u\|_{L^2} \leq C_S \|u\|_{L^2}$.

Let us assume by contradiction that $(T_k u) \wedge$ is not weakly convergent to $C_0 \hat{u}$. Then, there exists at least one $h_0(x) \in L^2$, so that $\int_{\mathbb{R}^n} (T_k u) \wedge (\xi) \bar{\tilde{h}}_0(\xi) d\xi$ is not convergent to $\int_{\mathbb{R}^n} C_0 \hat{u}(\xi) \bar{\tilde{h}}_0(\xi) d\xi$. This last fact implies the existence of an $\alpha > 0$ and of a subsequence $(T_{k_j})_1^\infty$, so that

$$\left| \int_{\mathbb{R}^n} (T_{k_j} u) \wedge (\xi) \bar{\tilde{h}}_0(\xi) d\xi - \int_{\mathbb{R}^n} C_0 \hat{u}(\xi) \bar{\tilde{h}}_0(\xi) d\xi \right| > \alpha > 0, \quad \forall j = 1, 2, \dots$$

On the other hand, because of the estimate $\|(T_{k_j} u) \wedge\|_{L^2} \leq C \|u\|_{L^2}$, we can extract another subsequence $T_{k_{j_\ell}}$ so that

$$\lim_{\ell \rightarrow \infty} (T_{k_{j_\ell}} u) \wedge = v \in L^2, \quad \text{weakly in } L^2.$$

We get in this way

$$\lim_{\ell \rightarrow \infty} (T_{k_{j_\ell}} u) \wedge (\xi) = C_0 \hat{u}(\xi) \quad \text{pointwise in } \mathbb{R}^n,$$

$$\lim_{\ell \rightarrow \infty} (T_{k_{j_\ell}} u) \wedge (\xi) = v(\xi) \in L^2, \quad \text{weakly in } L^2(\mathbb{R}^n).$$

Then, we prove that $C_0 \hat{u} = v$.

In fact, if we denote $\psi_\ell(\xi) = (T_{k_{j_\ell}} u) \wedge (\xi) - v(\xi)$, then $\psi_\ell(\xi) \rightarrow 0$ weakly in L^2 and $\psi_\ell(\xi) \rightarrow C_0 \hat{u}(\xi) - v(\xi)$ pointwise in $\mathbb{R}^n$. Taking real and imaginary parts we get (as is easily seen):

$$\operatorname{Re} \psi_\ell(\xi) \rightarrow 0, \quad L^2\text{-weakly}, \quad \operatorname{Re} \psi_\ell(\xi) \rightarrow \operatorname{Re}(C_0 \hat{u}(\xi) - v(\xi)) \quad \text{pointwise},$$

$\text{Im } \psi_\ell(\xi) \rightarrow 0$, L^2 -weakly, $\text{Im } \psi_\ell(\xi) \rightarrow \text{Im}(C_0 \hat{u}(\xi) - v(\xi))$ pointwise.

Hence we shall prove (see [1]) the following:

Proposition. Let $(\phi_n(\xi))_1^\infty$ be a sequence of real-valued $L^2(\mathbb{R}^n)$ -functions, so that

$$\phi_n(\xi) \rightarrow 0, \text{ weakly in } L^2,$$

$$\phi_n(\xi) \rightarrow \phi(\xi) \in L^2, \text{ pointwise in } \mathbb{R}^n.$$

Then $\phi(\xi) = 0$ almost-everywhere.

In fact, if $\phi(\xi)$ is not null almost-everywhere, we have $\text{meas}\{\xi, \phi(\xi) \neq 0\} > 0$. Let us denote:

$$E_n = \{\xi, \phi(\xi) \geq \frac{1}{n}\},$$

$$F_n = \{\xi, \phi(\xi) \leq -\frac{1}{n}\}.$$

$$\text{Then } \{\xi, \phi(\xi) \neq 0\} = \bigcup_1^\infty E_n \cup \bigcup_1^\infty F_n.$$

If all E_n, F_n were null-sets we would have $0 < \text{meas}\{\xi, \phi(\xi) \neq 0\} \leq \sum_1^\infty \text{meas}(E_n) + \sum_1^\infty \text{meas}(F_n) = 0$, a contradiction.

Hence, at least one E_n or one F_n is not a null-set, and we can say that there exists a set $S \subset \mathbb{R}^n$, $\text{meas } S > 0$, and $\phi(\xi) > C > 0$ on S or $\phi(\xi) < -C < 0$ on S .

Now the relation $\phi_n(\xi) \rightarrow \phi(\xi) \forall \xi \in S$ implies (in, say, the case where $\phi(\xi) > C > 0$) that $\forall \xi \in S$, $\exists j = j(\xi)$, so that $\phi_j(\xi) > C$ for $j > j(\xi)$. Let then $S_k = \{\xi \in S, j(\xi) = k\}$. It follows that $S = \bigcup_{k=1}^\infty S_k$ (easily seen). Hence $\exists \bar{k}$ so that $\text{meas}(S_{\bar{k}}) > 0$, and in this case it follows that

$$\phi_j(\xi) > C \quad \forall j \geq \bar{k}, \quad \forall \xi \in S_{\bar{k}}.$$

Let now $f(\xi) = 1$ on $S_{\bar{k}}^N$, $= 0$ outside $S_{\bar{k}}^N$ (1). Then $f \in L^2$ and

$$\int_{\mathbb{R}^n} \phi_j(\xi) f(\xi) d\xi = \int_{S_{\bar{k}}^N} \phi_j(\xi) d\xi > C \text{ meas}(S_{\bar{k}}^N), \text{ a contradiction which proves the}$$

proposition.

It follows that $C_0 \hat{u}(\xi) = v(\xi)$ almost everywhere and that

$$\lim_{\ell \rightarrow \infty} (T_{k_{j_\ell}} u)^\wedge = C_0 \hat{u} \quad \text{weakly in } L^2.$$

(1) Where $S_{\bar{k}}^N = \{\xi \in S_{\bar{k}}, |\xi| \leq N\}$; $\text{meas } S_{\bar{k}}^N$ is > 0 for N big enough.

This, however, is a contradiction with the previous relation

$$|\int_{\mathbb{R}^n} (T_{k_j} u)^\wedge(\xi) \bar{h}_0(\xi) d\xi - \int_{\mathbb{R}^n} C_0 \hat{u}(\xi) \bar{h}_0(\xi) d\xi| > \alpha > 0, \quad \forall j = 1, 2, \dots$$

So, our initial assumption that $(T_k u)^\wedge$ is not weakly convergent to $C_0 \hat{u}$ was wrong; we have proved that

$$\lim_{k \rightarrow \infty} (T_k u)^\wedge = C_0 \hat{u} \quad \text{weakly in } L^2, \quad \forall u \in S.$$

Our final step consists in proving that

$$\lim_{k \rightarrow \infty} (T_k u)^\wedge = C_0 \hat{u} \quad \text{weakly in } L^2, \quad \forall u \in L^2.$$

This is obtained in the following way:

First, $T_k u \rightarrow C_0 u$ weakly in L^2 iff $(T_k u)^\wedge \rightarrow C_0 \hat{u}$ weakly in L^2 . Hence $T_k u \rightarrow C_0 u$ weakly in L^2 , $\forall u \in S$. Now, for an arbitrary $u \in L^2$, $\forall h \in L^2$, and $\forall \varepsilon > 0$, let us take $u_\varepsilon \in S$, so that $\|u_\varepsilon - u\|_{L^2} < \varepsilon$. Then we have

$$\begin{aligned} |(T_k u - C_0 u, h)|_{L^2} &= |(T_k (u - u_\varepsilon) - C_0 (u - u_\varepsilon), h)|_{L^2} + |(T_k u_\varepsilon - C_0 u_\varepsilon, h)|_{L^2} \\ &\leq C_s \|u - u_\varepsilon\|_{L^2} \|h\|_{L^2} + |C_0| \|u - u_\varepsilon\|_{L^2} \|h\|_{L^2} + |(T_k u_\varepsilon - C_0 u_\varepsilon, h)|_{L^2} \\ &\leq \varepsilon (C_s \|h\|_{L^2} + |C_0| \|h\|_{L^2}) + |(T_k u_\varepsilon - C_0 u_\varepsilon, h)|_{L^2}. \end{aligned}$$

Remark now, that when $k \rightarrow \infty$, $(T_k u_\varepsilon - C_0 u_\varepsilon, h)_{L^2} \rightarrow 0$, because $u_\varepsilon \in S$. Hence, for $k \geq k_0(\varepsilon)$, we get the inequality

$$|(T_k u - C_0 u, h)|_{L^2} \leq \varepsilon (C_s + |C_0|) \|h\|_{L^2} + 1,$$

which proves our assertion and Theorem 4 too.

As a final application of the above results we shall demonstrate:

Theorem 5. If the inequality $\|Au\|_{H^s} \leq C_{s,\varepsilon} \|u\|_{H^{s-\varepsilon}} \quad \forall u \in H_s$ is true for some $\varepsilon > 0$, then

$$\max_{x \in \mathbb{R}^n} \overline{\lim}_{|\xi| \rightarrow \infty} |a(x, \xi)| = 0.$$

Proof. In fact, let us assume that for an $\varepsilon > 0$, the inequality

$$\|Au\|_{H^s} \leq C_{s,\varepsilon} \|u\|_{H^{s-\varepsilon}}, \quad \forall u \in H_s$$

is true. If $\max_{x \in \mathbb{R}^n} \overline{\lim}_{|\xi| \rightarrow \infty} |a(x, \xi)| > 0$, $\exists$ an $x_0 \in \mathbb{R}^n$ so that $\overline{\lim}_{|\xi| \rightarrow \infty} |a(x_0, \xi)| > 0$ too. Hence, $\exists (\xi_p)_{p=1}^\infty \rightarrow \infty$ so that $\lim_{p \rightarrow \infty} |a(x_0, \xi_p)| = d_0 > 0$. Also, $\{a(x_0, \xi_p)\}_{p=1}^\infty$ is a bounded sequence, hence a subsequence $\{a(x_0, \xi_{p_j})\}_{j=1}^\infty$ is convergent to $C_0 \in \mathbb{C}$.

Obviously, $|c_0| = d_0 > 0$ so that $c_0 \neq 0$. Consider now, $\forall u \in S$, the sequence $p_{x_0, \xi_p}^S u$ given by Theorems 3,4. $(\xi_p)_1^\infty$ is a subsequence of $(\xi_{p_j})_1^\infty$. Now,

$p_{x_0, \xi_p}^S u \in H_S$; so we shall have the estimate

$$\|AP_{x_0, \xi_p}^S u\|_{H^S} \leq c_{s, \varepsilon} \|p_{x_0, \xi_p}^S u\|_{H^{S-\varepsilon}}.$$

As we saw previously, as a corollary to Proposition 1, the relation

$$\lim_{p \rightarrow \infty} \|p_{x_0, \xi_p}^S u\|_{H^{S-\varepsilon}} = 0, \quad \forall u \in S$$

is satisfied. We get consequently that

$$\lim_{p \rightarrow \infty} \|AP_{x_0, \xi_p}^S u\|_{H^S} = 0, \quad \forall u \in S.$$

This implies, as $Q_{x_0, \xi_p} \in \mathcal{L}(H^S; H^0)$, that

$$\lim_{p \rightarrow \infty} \|Q_{x_0, \xi_p}^S AP_{x_0, \xi_p}^S u\|_{H^0} = 0, \quad \forall u \in S.$$

Hence, $Q_{x_0, \xi_p}^S AP_{x_0, \xi_p}^S u \rightarrow 0$ weakly in L^2 , $\forall u \in S$.

On the other hand, Theorem 4 gives

$$Q_{x_0, \xi_p}^S AP_{x_0, \xi_p}^S u \rightarrow c_0 u \quad \text{weakly in } L^2, \quad \forall u \in S.$$

So $c_0 u = 0 \quad \forall u \in S$, which contradicts $c_0 \neq 0$.

This proves Theorem 5.

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